

Final Exam

Answer all four questions, each in a *different* exam book.

Write your student number on each exam book and include your name on the book for Question 1.

Also write your name at the top of this question sheet, and please return this question sheet at the same time that you turn in the exam books.

Write legibly, show your work, and justify your responses.

1. [30 points] **Two Infinitely-Repeated Games in Oligopoly**

- (a) Consider the one-period Cournot game for n firms with constant marginal and average cost $c \in (0,1)$. The inverse demand curve in the market is described by $P(Q) = 1 - Q$, where $Q = \sum_{i=1}^n q_i$, where q_i is firm i 's choice of quantity produced. Denote this game as $\mathbf{G}_1$.
- (i) [3 points] Find the Nash equilibrium of $\mathbf{G}_1$. Calculate the market price P , the market quantity Q , and each firm's profit at the NE. (Hint: your answer should be in terms of n .)
- (ii) [4 points] Suppose that the n firms decide to collude so that the firms will jointly produce the monopoly quantity in this market, and each firm will produce the same quantity. Find the market quantity Q , the market price P , each firm's quantity q_i , and each firm's profit.

Now for parts (iii) and (iv) of (a), consider the infinitely-repeated version of $\mathbf{G}_1$, with discount factor $\delta \in (0,1)$.

- (iii) [3 points] Write down an n -tuple of trigger strategies that provide each firm with the same profits as calculated in part (ii). Specify the trigger strategies completely and precisely.
- (iv) [5 points] Determine the critical value $\hat{\delta}_1$ such that the trigger strategies of part (iii) form a subgame perfect equilibrium (SPE), i.e., find the lowest $\hat{\delta}_1$ such that the strategies are an SPE for all discount factors δ such that $\hat{\delta}_1 \leq \delta < 1$.
- (b) Consider the one-period Bertrand game for n firms with constant marginal and average cost $c \in (0,1)$. The demand curve in the market is described by $Q(P) = 1 - P$, where $p_i \in \mathbb{R}_+$ is firm i 's choice of price and $P = \min\{p_1, \dots, p_n\}$. The firm with lowest price takes the entire market. If multiple firms' prices are tied as the lowest, those firms will divide the market equally. Denote this game as $\mathbf{G}_2$.
- (i) [2 points] State the Nash equilibrium of $\mathbf{G}_2$, and each firm's profit at NE.
- Now for parts (ii) and (iii) of (b), consider the infinitely-repeated version of $\mathbf{G}_2$, with discount factor $\delta \in (0,1)$.
- (ii) [3 points] Write down an n -tuple of trigger strategies that provide each firm with the same profits as calculated in part (a)(ii). Specify the trigger strategies completely and precisely.
- (iii) [5 points] Determine the critical value $\hat{\delta}_2$ such that the trigger strategies of part (ii) form a subgame perfect equilibrium (SPE), i.e., find the lowest $\hat{\delta}_2$ such that the strategies are an SPE for all discount factors δ such that $\hat{\delta}_2 \leq \delta < 1$.
- (c) [5 points] Which critical value is greater: $\hat{\delta}_1$ or $\hat{\delta}_2$? What is the intuition behind this ranking? (In order to get full credit, you need to provide an intuitive explanation without depending solely on algebra.)

2. [20 points] **Should the staging chefs spread their wings?**

There are N staging chefs training at a Michelin-starred restaurant, and they are considering whether to leave to open their own restaurant. The staging chefs are N players in a static game of complete information. Every player simultaneously and independently selects from two possible actions: “leave to open a new restaurant together” or “stay in training”. The probability of success of the new restaurant is proportional to the number of players who leave to join it: specifically, the new restaurant succeeds with probability $\frac{L}{N}$, where L denotes the number of players who select “leave to open a new restaurant together”.

When a player selects “leave to open a new restaurant together”, she receives payoff of $a > 0$ if the new restaurant succeeds, while she loses her source of income and receives payoff of 0 if the new restaurant fails. When a player selects “stay in training”, she receives payoff of $b > 0$ regardless of the new restaurant’s success. The players are all risk-neutral and maximize their expected payoffs.

- (a) [4 points] Find all of the symmetric pure-strategy Nash equilibria of this game and describe any additional necessary conditions on a and b for such an equilibrium to exist.
- (b) [5 points] Provide a rigorous argument why an asymmetric Nash equilibrium in pure strategies cannot exist for this game.
- (c) [2 points] Suppose that $b > a$. Provide a rigorous argument that there is a unique Nash equilibrium (symmetric or asymmetric, and in pure or mixed strategies).
- (d) [2 points] Suppose that $a > Nb$. Provide a rigorous argument that there is a unique Nash equilibrium (symmetric or asymmetric, and in pure or mixed strategies).
- (e) [3 points] Provide a rigorous argument why any Nash equilibrium in which every chef nontrivially randomizes must be a symmetric equilibrium (i.e., each chef selects “leave to open a new restaurant together” with probability p and selects “stay in training” with probability $1 - p$, where every chef randomizes using the same probability p , where $p \in (0,1)$).
- (f) [4 points] Finally, suppose that $Nb \geq a \geq b$. Determine all of the symmetric Nash equilibria of this game.

3. [30 points] **Two “Half-Baked” Auction Games**

- (a) [15 points] Consider the following sealed-bid auction for a single item. There are **three** risk-neutral bidders, whose utilities are quasilinear in money. Bidder i ’s valuation, denoted by v_i , for the item is private information and the v_i ($i = 1, 2, 3$) are drawn from i.i.d. random variables that are each uniformly distributed on the interval $[0,1]$. After observing her own valuation v_i , each bidder i simultaneously and independently submits a sealed bid $b_i \in [0,1]$ for the item. The highest bidder wins the item. However:

- The highest bidder pays the amount of her bid; and
- Each losing bidder pays **one-half** the amount of her bid.

For example, if $b_i > b_j > b_k$, then only bidder i wins the item; but bidder i pays b_i , bidder j pays $\frac{1}{2} b_j$, and bidder k pays $\frac{1}{2} b_k$. Ties are resolved by a fair randomization.

Solve for a symmetric, Bayesian-Nash equilibrium of this game. Explain and fully justify your work.

- (b) [15 points] Consider another, slightly unusual, sealed-bid auction for a single item. There are two risk-neutral bidders who have valuations which are private information and which are drawn from i.i.d. random variables that are uniformly distributed on the interval $[0,1]$. There are only two allowable bids. After observing her own valuation, each of the two bidders simultaneously and independently submits a bid selected from the two-element set $\{0, 1/3\}$ (i.e., the only allowable bids are 0 and $1/3$). The highest bidder wins the item. However:

- The highest bidder pays half of her bid to the seller; and
- The highest bidder pays the other half of her bid to the other bidder.

So, the “losing” bidder does not receive the item, but she receives half of the winning bidder’s payment. In the event of a tie, each bidder takes the role of the winning bidder with probability $1/2$, and takes the role of the losing bidder with probability $1/2$.

Solve for the Bayesian-Nash equilibrium of this game. Explain and fully justify your work.

4. [20 points] **A sequential bargaining game**

Consider a sequential bargaining game between a seller and a buyer. The seller owns a good, for which the seller’s valuation equals 0 and the buyer’s valuation equals 30. Each player applies a discount factor $\delta \in (0,1)$ between successive periods. In period one, a die is rolled. (A die is a cube with the numbers “1”, “2”, “3”, “4”, “5” and “6” on its six faces; when the die is rolled, the six numbers come up with equal probabilities.)

- When “1” or “2” is rolled, the seller is chosen to make an offer p_S , where $p_S \in [0, 30]$;
- When “3” or “4” is rolled, the buyer is chosen to make an offer p_B , where $p_B \in [0, 30]$; and
- When “5” or “6” is rolled, neither player is chosen to make an offer, and the game immediately continues to the next period (and payoffs in the next period are discounted).

When one player makes an offer, the other player can immediately accept or reject this offer. If he accepts, the game ends and quasilinear utilities accrue without discounting; while if he rejects, the game continues to period two. In period two, payoffs are discounted by δ . The die is again rolled, independently of previous periods, determining by the same rules which player (or if neither player) is chosen to make an offer. If the player receiving the offer accepts, the game ends and quasilinear utilities accrue without additional discounting; and if he rejects, the game continues to period three, etc. The process continues until agreement is reached.

- (a) [4 points] Let U_S denote the seller’s equilibrium expected payoff evaluated at the start of the game and let U_B denote the buyer’s equilibrium expected payoff evaluated at the start of the game. Provide a short but precise argument why $U_S = U_B$.
- (b) [16 points] Determine the subgame-perfect equilibrium of this game. In particular, if the seller is chosen to make the first offer, what price does she offer? If the buyer is chosen to make the first offer, what price does he offer? And what is the equilibrium expected payoff of each player?