

Final Exam

Answer all four questions, each in a *different* exam book.

Write your student number on each exam book and include your name on the book for Question 1.

Also write your name at the top of this question sheet, and please return this question sheet at the same time that you turn in the exam books.

Write legibly, show your work, and justify your responses.

1. [20 points] **A voting model with advertising**

Consider the following voting game with two candidates, A and B. Locations on the political spectrum are denoted by the variable x , where $x \in [0, 1]$. Voters are uniformly distributed on $[0, 1]$. The two candidates simultaneously and independently locate themselves at positions $x_A \in [0, 1]$ and $x_B \in [0, 1]$, respectively, and they spend $c_A \in [0, 1]$ and $c_B \in [0, 1]$ in order to advertise their respective campaigns.

The unusual feature of this game is that the voters at locations within the subinterval $[0, 0.2]$ vote as a block for the same candidate and their decision depends solely on the advertising spending of the two candidates. More precisely:

All the voters $x \in [0, 0.2]$ vote for candidate A with probability $\left(\frac{c_A}{c_A + c_B}\right)$, and

All the voters $x \in [0, 0.2]$ vote for candidate B with probability $\left(\frac{c_B}{c_A + c_B}\right)$.

If $c_A = 0 = c_B$ then the respective probabilities are $\frac{1}{2}$ and $\frac{1}{2}$.

For the avoidance of doubt, all voters $x \in [0, 0.2]$ vote for the same candidate!

More conventionally, each voter at location $x \in [0.2, 1]$ votes for the candidate closest to her. If the two candidates select the same position, then exactly half of the voters in subinterval $[0.2, 1]$ vote for each of the candidates.

Candidate i receives a payoff of $(1 - c_i)$ if he obtains greater than 50% of the vote count, and he receives a payoff of $(-c_i)$ if he obtains less than 50% of the vote count. (He receives a payoff of $(\frac{1}{2} - c_i)$ if he obtains exactly 50% of the vote count.) Both candidates are risk neutral and maximize their expected payoffs.

Solve for all of the Nash equilibria in pure strategies of this game. Be sure to demonstrate that there are no other possible pure-strategy Nash equilibria other than those that you identify.

2. [20 points] **An auction game with discrete bids**

Consider the following sealed-bid auction game for a single item. There are two risk-neutral bidders, named Bidder 1 and Bidder 2, whose valuations are drawn from independent random variables. Bidder 1's valuation is uniformly distributed on the interval $[0, 1]$. **However, Bidder 2's valuation is uniformly distributed on the interval $[0, \frac{1}{2}]$.** After observing its own valuation for the item, each of the two bidders simultaneously and independently submits a bid from the two-element set $\{0, \frac{1}{4}\}$. That is, each bidder is only allowed to submit a bid of zero or one-fourth. The high bidder wins the item and pays the amount of its bid (i.e., it is a first-price auction). Ties are resolved by the toss of a fair coin.

Solve for the Bayesian-Nash equilibrium of this first-price auction, fully explaining your work.

3. [30 points] **Common-value auctions with private signals**

Consider the following sealed-bid auction games for a single item. There are two risk-neutral bidders, identified as bidder 1 and bidder 2, who receive independent private signals about the value of the item. Each bidder receives a signal of high (H), with probability θ , or low (L), with probability $(1 - \theta)$. The bidders' signals are drawn independently. Bidder 1's value for the good is v_H if and only if bidder 1's signal is H **and** bidder 2's signal is H . Otherwise, bidder 1's value is v_L , where $0 < v_L < v_H$. Similarly, bidder 2's value for the good is v_H if and only if bidder 1's signal is H **and** bidder 2's signal is H . Otherwise, bidder 2's value is v_L .

There is a sealed-bid auction. Each bidder may submit any bid in the interval $[0, v_H]$. The bidders submit their bids simultaneously and independently. The high bidder wins the item. Ties are resolved by the toss of a fair coin. The prices paid by the winner differ in (a) and (b).

- (a) Suppose that the game is a second-price auction (i.e., the winning bidder pays the amount of the losing bid). Determine whether this auction game has a symmetric Bayesian-Nash equilibrium in pure strategies. If it does, calculate the symmetric pure-strategy Bayesian-Nash equilibrium and show that there are no profitable deviations. If it does not, prove that a symmetric pure-strategy Bayesian-Nash equilibrium of the game does not exist.
- (b) Instead, suppose that the game is a first-price auction (i.e., the winning bidder pays the amount of its winning bid). Determine whether this auction game has a symmetric Bayesian-Nash equilibrium in pure strategies. If it does, calculate the symmetric pure-strategy Bayesian-Nash equilibrium and show that there are no profitable deviations. If it does not, prove that a symmetric pure-strategy Bayesian-Nash equilibrium of the game does not exist.

4. [30 points] **Contribution games**

Begin by considering the static game of complete information, \mathbf{G} , in which Alice and Bob simultaneously choose contributions $a \in [0, 1]$ and $b \in [0, 1]$, respectively, and receive payoffs of $U_A = 2b - a$ and $U_B = 2a - b$, respectively.

- (a) Find all of the Nash equilibria of \mathbf{G} , justifying your answer.
- (b) Next, consider the infinitely-repeated game based upon \mathbf{G} , where each player has discount factor $\delta \in (0, 1)$. Determine the "Folk Theorem region" (i.e., the set of points in payoff space that satisfy the hypothesis of the Folk Theorem).
- (c) For every contribution pair (a^*, b^*) corresponding to a point in the Folk Theorem region, determine the critical discount factor $\delta^* \in (0, 1)$ such that for any $\delta \in [\delta^*, 1)$, there exists a trigger-strategy equilibrium of the infinitely-repeated game based upon \mathbf{G} in which Alice and Bob contribute a^* and b^* , respectively, on the equilibrium path.
- (d) Finally, consider an alternating-offer bargaining game between Alice and Bob. When it is a player's turn to make an offer, the player proposes a contribution pair (a^*, b^*) for the two players. If the other player accepts, these contributions are implemented on behalf of the players in a binding way and the game ends. If the other player rejects, they progress to the next period, in which the other player gets to make an offer. Alice makes the offer and Bob accepts or rejects in the initial period 0. In subsequent periods, the offers and accept/reject decisions alternate. Each player has discount factor $\delta \in (0, 1)$, so if the players reach agreement in period t , their payoffs are $\delta^t U_A$ and $\delta^t U_B$, respectively. Determine a subgame-perfect equilibrium of this alternating-offer bargaining game, being sure to specify the strategies completely and to show that it is a subgame-perfect equilibrium.