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One is almost enough for monopoly

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It has been argued that two factors—product durability and (potential) entry—may force a monopolist to price at marginal cost. This article shows that when these two forces coexist, the tendency toward competition may be negated. First, we prove that durable goods oligopolists *without commitment powers may attain joint profits arbitrarily close to those of a monopolist with perfect commitment power. Second, we demonstrate that the presence of a potential entrant may enable a durable goods* monopolist *to act as if he had commitment power. Thus, potential as well as actual entry may restore monopoly power.*

1. Introduction

■ Classic economic analysis suggests that a monopolist has the ability to exercise market power: to charge greater than the competitive price and, often, to earn supranormal profits. Adam Smith was certainly not the first to make this observation when, in *The Wealth of Nations,* he wrote, "The monopolists, by keeping the market constantly understocked, by never fully supplying the effectual demand, sell their commodities much above the natural price, and raise their emoluments, whether they consist in wages or profits, greatly above their natural rate." Nor was he the last; most economics texts today faithfully recite a similar story.

Recently, two strands of research have called into question the conventional wisdom about monopoly. The first, introduced by Coase (1972), considers the problem of the durable goods monopolist. Coase offered the following intuition. Suppose that a monopolist were to offer a durable good for sale at the static monopoly price, and suppose that all consumers who valued the good at greater than the static monopoly price were to purchase it. Then, if the static monopoly price exceeded the marginal cost of the good, the monopolist would have every incentive to cut the price of the good to generate additional sales; moreover, the process would continue until price equalled marginal cost. Consumers, anticipating this price-slashing behavior, would choose to postpone purchasing the good when faced with

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the static monopoly price. Rational consumer behavior forces the profit-maximizing monopolist to introduce the good at close to marginal cost.

Bulow (1982) analyzed Coase's reasoning in a finite-horizon model. In the final period a monopolist who lacks commitment power charges the static monopoly rental price for the residual demand curve. By backward induction Bulow calculates the monopolist's best action in each earlier period and shows that it is always to charge unambiguously less than the static monopoly price.

Stokey (1979, 1981) formalized and proved Coase's intuition. In the first article she demonstrated that a monopolist who can make binding commitments about future sales does best by introducing the good at the static monopoly price and never cutting the price afterwards. In other words, the best intertemporal price discrimination is no intertemporal price discrimination. In the second article Stokey considered the problem of the durable goods monopolist who lacks commitment powers. She constructed an equilibrium of the infinite-horizon model, which is the limit of the unique equilibria of finite-horizon versions of the same model. The price path associated with that equilibrium (including the initial price) converges to marginal cost as the length of each period goes to zero; this proposition is commonly known as the "Coase conjecture."[1]

Gul, Sonnenschein, and Wilson (1986) modelled this situation as an infinite-horizon game between a single firm and a continuum of consumers. The notion of "no commitment" is captured by the requirement of subgame perfection of the equilibrium. These authors discovered a continuum of additional subgame-perfect equilibria in this game, but proved that an interesting subclass (weak Markov equilibria) behave in the manner of Stokey's backward induction equilibrium—they also satisfy the Coase conjecture.

The second strand of literature challenging the common wisdom about monopoly was introduced by Baumol, Bailey, Panzar, and Willig. (See, for example, Baumol *et al.*, (1982).) These researchers focus on the role of potential, costlessly reversible entry in determining the monopoly outcome. In essence, they seize upon the result of Bertrand that "two is enough for competition"—under constant marginal (and average) cost, the unique duopoly Nash equilibrium in price strategies is for both firms to engage in marginal-cost pricing. The contestable markets literature goes further by arguing that under certain conditions of monopoly and entry (most notably, zero sunk cost), "one is almost enough for competition."

In this article we attempt to combine and extend the reasoning of the Coase conjecture and contestable markets. We prove that for durable goods oligopoly, as the time interval between successive offers approaches zero, *all* joint payoffs between zero and static monopoly profits are attainable. Moreover, we show that actual entry into the market is unnecessary to obtain this result: a durable goods monopoly with a potential entrant, who in equilibrium never enters, may also earn static monopoly profits. We further find the full set of equilibrium joint profits associated with arbitrary discount factors. Our results indicate that even when firms discount the future significantly, effective collusion can still occur.

Closely related to our article is Gul's "Noncooperative Collusion in Durable Goods Oligopoly," which also appears in this issue of *The RAND Journal of Economics*.[2] Many of the results in these two articles are quite similar. Gul, however, emphasizes the limiting result for more general demand curves and treats the case of asymmetric market shares. We provide a precise description of optimally collusive equilibria for a parameterized family of demand curves for all discount factors and stress the model of potential entry. Thus, the two articles are complementary.

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[1] Kahn (1986) considered the durable goods monopolist with increasing marginal cost; Bond and Samuelson (1984) examined a durable good subject to depreciation. Both found the Coase conjecture modified.

[2] We read an earlier version of Gul's article, entitled "Foundations of Dynamic Oligopoly," after completing the working paper precursor of this article.

In Section 2 we describe the model and establish two theorems on subgame-perfect equilibria. Section 3 and Appendix A characterize optimally collusive and minimally collusive subgame-perfect equilibria. In Section 4 we identify the entire set of joint profits supported by subgame-perfect equilibria for all discount factors δ, and we indicate its limiting behavior as δ approaches one. We establish analogous results for monopoly with potential entry in Section 5 and present some details in Appendix B. In the Conclusion we compare the present results with those of a sequel paper (Ausubel and Deneckere, 1986), which studies durable goods monopoly.

2. A durable goods oligopoly

■ It is reasonable to suppose that an oligopoly behaves more competitively than a monopoly. Hence, if one believed that profits in a durable goods monopoly were drastically curtailed by the market forces depicted in the Coase conjecture, one would probably also suspect the same of durable goods oligopoly.[3] After all, how could the presence of a rival make the monopolist *better* off? To understand why this intuitive argument is misleading, it is instructive to reflect on the reason a monopolist may be forced to act competitively. The problem is essentially this: once he has sold an initial quantity of the good, he will find it tempting to sell some additional output as long as his accumulated output sold remains below the competitive level. If there is virtually no restraint on the speed at which the monopolist can sell additional units (if the time interval between successive offerings is very small), the market will almost immediately be saturated with the competitive output. Rational consumers will foresee this, and purchase the good at no more than the competitive price. It is this inability of the monopolist to control the speed at which he sells, or put somewhat differently, his inability to punish his future types for deviant behavior,[4] that impairs his monopoly power. The monopolist may thus be better off having a competitor around to punish him (or his future types) for selling the good more quickly than is optimal.

We consider a market for a good that is infinitely durable and demanded only in quantity zero or one. Consumers, who are infinitely lived, have reservation prices for the good that are distributed over the interval $V = [0, 1]$ according to the distribution function $F(v) = v^{\alpha}$ $(0 < \alpha < \infty)$. Because they discount future utility at the interest rate r, individuals who have a reservation value of v and who obtain the good at time t for the price p_t derive a net surplus given by

$$e^{-rt}[v - p_t].$$

Two producers serve this market, which is open at discrete times spaced z apart $(t = 0, z, 2z, \ldots, nz, \ldots)$. The timing of moves within the period is as follows. Firms first name their prices. Consumers (who did not buy in previous periods) then decide whether to purchase the good in the current period. We shall sometimes refer to the "period" n rather than to the time t $(= nz)$. There is no constraint on the amount of output any producer can supply at a given date, and production occurs at a common marginal (and average) cost of c, which we assume for convenience is zero. Firms are interested in maximizing the net present value of profits, discounted at the interest rate r.

A *strategy* for producer i specifies the price he will charge at each moment in time as a function of the history of prices charged by both competitors and the history of purchases by consumers. A strategy for a consumer specifies, given the current price charged and the history of past prices and purchases, whether to buy the good in the current period.

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[3] Coase (1972, p. 144) did. He wrote, "With complete durability, the price becomes *independent of the number of suppliers* and is thus always equal to the competitive price" (emphasis added).

[4] Once a time period has lapsed, the original monopolist transforms into a greedy type who wants to maximize profits from then on rather than to implement the sales plan to which his previous type would have liked to commit him.

Let $G(z, r)$ denote the game when the time interval between successive sales is z and payoffs are discounted at the rate r. Let σ_i be a *pure* strategy for producer i ($i = 1, 2$) so that σ_i is a sequence of functions $\{\sigma_i(n)\}_{n=0}^{\infty}$. The function $\sigma_i(n)$ at date nz determines i's price in period n as a function of the prices charged by both competitors in previous periods and the actions chosen by consumers in the past. The latter history is conveniently summarized by the set $V_n = \{v$: consumer v did not buy at any time $t < nz\}$.[5] We shall impose measurability restrictions on joint consumer strategies that imply that the set V_n is measurable, i.e., $V_n \in \Omega$, where Ω is the Borel σ-algebra on V. Then $\sigma_i(n)$: $S^{n-1} \times \Omega \rightarrow S_i$, with $S_i = [0, 1]$ for $i = 1, 2$ and $S = S_1 \times S_2$. A strategy combination for consumers is a sequence of functions $\{\tau^n\}_{n=0}^{\infty}$, where τ^n: $S^n \times \Omega \times V \rightarrow \{0, 1, 2\}$ is such that for each $s^n \in S^n$ and each $B \in \Omega$, $\tau^n(s^n, B, \cdot)$ is measurable. Decision "0" here is to be interpreted as "do not buy from either supplier." The decision "i" ($i = 1, 2$) means "buy from supplier i." Without loss of generality, we may assume that τ^n is zero for any consumer v for whom $\tau^j > 0$ for some period before n.

Let Σ_i be the strategy space for player i ($i = 1, 2$) and Υ be the set of strategy combinations for consumers. Furthermore, let $\sigma = (\sigma_1, \sigma_2)$ and $\tau = \{\tau^n\}_{n=0}^{\infty}$. The strategy profile $\{\sigma, \tau\}$ generates a path of prices and sales that can be computed recursively. The pattern of prices and sales over time, in turn, determines the payoff to the players. Let $\pi^i(\sigma, \tau)$ be the discounted present value of the profits of firm i, generated by the strategy profiles (σ, τ), and let $u^v(\sigma, \tau)$ be the discounted net surplus derived by consumer v. The profile (σ, τ) is a Nash equilibrium if

$$\pi^i(\sigma_i, \sigma_{-i}, \tau) \geqslant \pi^i(\sigma'_i, \sigma_{-i}, \tau) \qquad \forall \sigma'_i \in \Sigma_i, \qquad \forall i,$$

and

$$u^v(\sigma, \tau_v, \tau_{-v}) \geqslant u^v(\sigma, \tau'_v, \tau_{-v}) \qquad \forall \tau'_v \in \Upsilon_v, \qquad \forall v \in V,$$

where τ_v is the projection of τ onto the vth coordinate (and similarly for Υ_v), and τ_{-v} is the projection of τ onto all coordinates except the vth one. An n-period history of the game is a sequence of prices for each firm in periods $0, \ldots, (n - 1)$ and a specification of the set of consumers who did not buy in any period before n. We denote such a history by the symbol H_n. The symbol H'_n refers to H_n followed by prices announced by both firms in period n. The strategy profile (σ, τ) induces strategy profiles $(\sigma|_{H_n}, \tau|_{H_n})$ and $(\sigma|_{H'_n}, \tau|_{H'_n})$ after the histories H_n and H'_n, respectively. With this notation (σ, τ) is a *subgame-perfect* equilibrium if and only if (σ, τ) is a Nash equilibrium and $(\sigma|_{H_n}, \tau|_{H_n})$ is a Nash equilibrium in the game remaining after the history H_n, for all H_n and for all n, and similarly after any history H'_n.

Denote the price charged by firm i in period n along the outcome path generated by (σ, τ) as $p^i_n(\sigma, \tau)$, and let $p_n(\sigma, \tau) = \min_i p^i_n(\sigma, \tau)$. In our first theorem we prove that a necessary condition for subgame perfection is that prices be set so that *all* customers (with valuations exceeding marginal cost) are eventually induced to purchase. Otherwise, the profits from following the equilibrium approach zero, while the profits from deviating remain positive.[6]

Theorem 1. In any subgame-perfect equilibrium of $G(z, r)$, $\inf_n p_n(\sigma, \tau) = 0$.

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[5] To capture the idea that customers are anonymous and nonatomic, we shall assume that plays that result in the same sequence of prices and the same measure of consumer acceptances yield identical strategy choices for both the monopolist and the consumers in subsequent periods.

[6] If, instead, consumer valuations are bounded away from zero (see the last paragraph of Section 4), Theorem 1 becomes: In any subgame-perfect equilibrium, either $\inf_n p_n = \underline{v}$ or $p_n = 0$ (for all $n \geqslant 0$). The result $\inf_n p_n \leqslant \underline{v}$ follows analogously to Theorem 1. If $p_m < \underline{v}$ for some $m \geqslant 0$, it is easy to show that a consumer with valuation $\underline{v}$ purchases in finite time, which implies the Bertrand equilibrium.

Proof. Suppose to the contrary that $\hat{p} = \inf_n p_n(\sigma, \tau) > 0$. We shall show that there exists an n sufficiently large that each firm has an incentive to deviate from its strategy σ_i.

First, we calculate a *lower bound* on the net present value of profits generated by a deviation in period n consisting of charging a price $p < p_n$ (for some firm $i = 1, 2$). Such a deviation will be least profitable when it leads to maximally competitive behavior in future periods—that is, when both firms charge a zero price in the next period. In that case a deviation engenders zero profits in subsequent periods and makes consumers extremely averse to buying in period n. The profits from deviating in period n can be bounded below by assuming that all customers with valuations greater than $\hat{p}$ have already been served. By charging a price $p \leqslant \hat{p}$, our deviant will attract all customers with valuation exceeding $\tilde{v}$ such that $\tilde{v} - p \geqslant e^{-rz}\tilde{v}$. Letting $\delta = e^{-rz}$, we have $\tilde{v} \geqslant p/(1 - \delta)$. Indeed, those customers will find it attractive to buy in period n at the price p rather than to wait until period $(n + 1)$ for a zero price.

Recall that the number of customers with valuation less than v is given by v^α. By setting a price p, then, profits in period n are at least

$$\left[\hat{p}^\alpha - \left(\frac{p}{1-\delta}\right)^\alpha\right] p.$$

This expression is maximized at $p = \rho(1 - \delta)\hat{p}$, where $\rho = (1 + \alpha)^{-1/\alpha}$, with a corresponding profit of $\mu(1 - \delta)\hat{p}^{(1+\alpha)}$, where $\mu = \alpha\rho/(1 + \alpha)$. Thus, a firm will certainly choose to deviate if for some n

$$\mu(1-\delta)\hat{p}^{(1+\alpha)} > \pi_n(\sigma, \tau),$$

where $\pi_n(\sigma, \tau)$ is the net present value (evaluated in period n) to a single firm in the subgame starting after the equilibrium history H_n. It is clear that $\pi_n(\sigma, \tau)$ approaches 0 as n approaches ∞. For large enough n either $p_n = \hat{p}$ or p_n gets arbitrarily close to $\hat{p}$, so that the set of customers who remain to be served becomes arbitrarily small. Thus, the inequality above will be satisfied for large enough n unless $\hat{p} = 0$. *Q.E.D.*

Observe that ρ $(=[1 + \alpha]^{-1/\alpha})$, defined in the proof above, is the static monopoly price when $F(v) = v^\alpha$, and μ $(=\rho[1 - \rho^\alpha])$ equals the static monopoly profits.

The *worst* subgame-perfect equilibrium payoff from the sellers' viewpoint is easily seen to be equal to zero: it is implemented by the strategies $\sigma_i(n) = 0$ and $\tau_v^0 = 1$ or 2. In fact, this payoff coincides with the sellers' minimax payoff. This equilibrium will be useful as a "punishment" regime following deviations from more collusive price paths, as in Friedman (1971), Aumann and Shapley (1976), and Abreu (1985). Next, we derive *necessary and sufficient* conditions that any subgame-perfect price path $\{p_n\}$ must satisfy. (We suppress the dependence of p_n and π_n on the equilibrium strategies (σ, τ) whenever no confusion is likely to arise.) Let $v_n = \sup\{v\colon \tau_v^j = 0 \text{ for all } j \leqslant n - 1\}$, so that v_n is the highest consumer valuation remaining as we enter period n. Observe that along the equilibrium path the single number v_n summarizes the actions chosen by consumers in periods before n. Indeed, consumer optimization implies that whenever some consumer v has bought before period n, any consumer with valuation $v' > v$ should also have bought before n.

Theorem 2 states that for every period the net present value of profits along the equilibrium path for each firm must exceed the profits from optimally deviating.

Theorem 2. Any subgame-perfect price path $\{p_n\}$ of $G(z, r)$ must satisfy:

$$\pi_n^i \geqslant \mu(1-\delta)v_n^{1+\alpha} \qquad \text{for all } n \text{ such that} \qquad p_n \geqslant \rho(1-\delta)v_n \tag{1}$$

$$\pi_n^i \geqslant p_n\left[v_n^\alpha - \left(\frac{p_n}{1-\delta}\right)^\alpha\right] \qquad \text{for all } n \text{ such that} \qquad p_n < \rho(1-\delta)v_n. \tag{2}$$

Conversely, any price path $\{p_n\}$ that satisfies (1) and (2) is an equilibrium price path.

Proof. Suppose that inequality (1) is not satisfied for some n where $p_n > \rho(1-\delta)v_n$. By setting the price $p = \rho(1-\delta)v_n$ at date n, and thereby undercutting p_n, a firm will earn *at least* $\mu(1-\delta)v_n^{1+\alpha}$ in immediate profits in period n. Since this exceeds π_n, any firm has an incentive to deviate from $\{p_n\}$. Similarly, suppose that inequality (2) is not satisfied for some n, where $p_n < \rho(1-\delta)v_n$. Then by charging a price slightly below p_n, any firm can earn *at least* profits arbitrarily close to $\left[v_n^\alpha - \left(\frac{p_n}{1-\delta}\right)^\alpha\right]p_n$. This again contradicts the hypothesis that $\{p_n\}$ is a subgame-perfect price path.

Conversely, any price path $\{p_n\}$ satisfying (1) and (2) can be supported by the threat and corresponding consumer expectations of reversion to the subgame-perfect equilibrium that involves pricing at marginal cost as soon as any firm deviates.[7] *Q.E.D.*

3. Optimally collusive subgame-perfect equilibria

■ In general, a multiplicity of subgame-perfect equilibria satisfy the necessary and sufficient conditions of Theorem 2. In this section we characterize the optimally collusive equilibrium, i.e., the subgame-perfect equilibrium that maximizes joint profits. We also determine the minimal (nonzero)[8] joint profits associated with any subgame-perfect equilibrium. To keep this characterization as simple as possible, we shall assume—for reasons extraneous to the model—an "equal division rule": in any period in which both firms charge the same price, an equal fraction of consumers chooses to patronize each firm.[9]

The potentially formidable task of characterizing the optimally (and minimally) collusive subgame-perfect equilibria is greatly simplified by Theorems A1 and A2 of Appendix A. These theorems demonstrate that there is no loss of generality in restricting attention to *simple-strategy profiles.* Such profiles are completely described by a pair of numbers (p_0, ϵ), where p_0 is the price to be charged in period 0, and ϵ denotes the fraction by which firms lower prices in each subsequent period. The equilibrium price sequence corresponding to the simple-strategy profile (p_0, ϵ) is thus $p_n = p_0\epsilon^n$. This price sequence implies a pattern of historic variables v_n of

$$v_n = \frac{p_{n-1} - \delta p_n}{(1-\delta)} = \frac{(1-\delta\epsilon)}{(1-\delta)}p_{n-1} \equiv \gamma p_{n-1}, \qquad n \geqslant 1.$$

Since v_n is strictly decreasing in n as long as $\epsilon < 1$, a simple-strategy profile implies sales in all periods if $v_1 = \gamma p_0 < 1$. We can calculate sales in period n to be

$$v_n^\alpha - v_{n+1}^\alpha = (1-\epsilon^\alpha)\gamma^\alpha p_{n-1}^\alpha$$

for $n \geqslant 1$, provided $p_0 < \gamma^{-1}$. We can now compute π_n along the sequence $p_n = p_0\epsilon^n$ (with $p_0 < \gamma^{-1}$):

$$\pi_n = \frac{1}{2}\sum_{k=0}^{\infty}\delta^k p_{n+k}(v_{n+k}^\alpha - v_{n+k+1}^\alpha) = \frac{1}{2}\,\frac{\gamma^\alpha\epsilon(1-\epsilon^\alpha)}{1-\delta\epsilon^{1+\alpha}}\,p_{n-1}^{1+\alpha}$$

[7] Off-equilibrium-path behavior on the part of consumers can be treated in the following way: If a set of measure zero of consumers deviates, those deviations are ignored. If a set of positive measure deviates, firms revert to pricing at marginal cost in all future periods, and the remaining consumers' strategies are optimal subject to this prediction.

[8] The lowest joint profits in a subgame-perfect equilibrium are zero. We seek the next higher profits attainable.

[9] As shown in Appendix A, the equal division rule implies that there is no loss of generality in further focusing attention on *symmetric* equilibria, i.e., subgame-perfect equilibria with the property that $\sigma_1 = \sigma_2$.

for $n \geqslant 1$, and for $n = 0$,

$$\pi_0 = \frac{1}{2}[1 - \gamma^\alpha p_0^\alpha]p_0 + \frac{\delta}{2}\frac{\gamma^\alpha \epsilon(1-\epsilon^\alpha)}{1-\delta\epsilon^{1+\alpha}} p_0^{1+\alpha}.$$

To complete the description of simple-strategy profiles, we still have to describe the behavior of firms off the equilibrium path. We let deviations from the path $p_n = p_0\epsilon^n$ be punished by both firms' reverting to the *worst* subgame-perfect equilibrium, i.e., pricing at marginal cost in every future period.

We can immediately rule out any simple profile (p_0, ϵ) for which $p_0 \geqslant \gamma^{-1}$ as potentially optimally collusive; it is dominated by the simple profile $(\epsilon p_0, \epsilon)$.[10] In our quest for optimal simple-strategy profiles, the following lemma will be useful.

Lemma 1. A simple-strategy profile (p_0, ϵ) with $\rho(1 - \delta) \leqslant p_0 < \gamma^{-1}$ is a subgame-perfect equilibrium if and only if $\epsilon \geqslant \rho/(1 + \rho\delta)$,

$$\text{(i)} \quad \frac{\epsilon(1-\epsilon^\alpha)}{1-\delta\epsilon^{1+\alpha}} \geqslant 2\mu(1-\delta\epsilon), \qquad \text{and}$$

$$\text{(ii)} \quad p_0 - \gamma^\alpha p_0^{1+\alpha}\left[\frac{1-\delta\epsilon}{1-\delta\epsilon^{1+\alpha}}\right] \geqslant 2\mu(1-\delta).$$

If $0 < p_0 < \rho(1 - \delta)$, the simple-strategy profile (p_0, ϵ) is a subgame-perfect equilibrium if and only if $\epsilon \geqslant \rho/(1 + \rho\delta)$, (i) holds, and

$$\text{(iii)} \quad p_0 - \gamma^\alpha p_0^{1+\alpha}\left[\frac{1-\delta\epsilon}{1-\delta\epsilon^{1+\alpha}}\right] \geqslant 2p_0\left[1 - \left(\frac{p_0}{1-\delta}\right)^\alpha\right].$$

Proof. The condition $\epsilon \geqslant \rho/(1 + \rho\delta)$ implies that $p_n \geqslant \rho(1 - \delta)v_n$ for $n \geqslant 1$. If $p_0 \geqslant \rho(1 - \delta)$, Theorem 2 implies that (p_0, ϵ) is a subgame-perfect equilibrium if and only if

$$\pi_n = \frac{\gamma^\alpha \epsilon(1-\epsilon^\alpha)}{2(1-\delta\epsilon^{1+\alpha})} p_{n-1}^{1+\alpha} \geqslant \mu(1-\delta)v_n^{1+\alpha} = \mu(1-\delta)\gamma^{1+\alpha}p_{n-1}^{1+\alpha}$$

$$\pi_0 = \frac{1}{2}\left\{p_0 - \gamma^\alpha p_0^{1+\alpha}\left(1 - \frac{\delta\epsilon(1-\epsilon^\alpha)}{1-\delta\epsilon^{1+\alpha}}\right)\right\} \geqslant \mu(1-\delta).$$

Simple algebraic manipulations then yield (i) and (ii). If $p_0 < \rho(1 - \delta)$, inequality (2) of Theorem 2 is applicable for $n = 0$ and yields (iii) in place of (ii).

If $\epsilon < \rho/(1 + \rho\delta)$, $p_n < \rho(1 - \delta)v_n$, and Theorem 2 now requires that

$$\pi_n = \frac{\gamma^\alpha \epsilon(1-\epsilon^\alpha)}{2(1-\delta\epsilon^{1+\alpha})} p_{n-1}^{1+\alpha} \geqslant \left[\gamma^\alpha - \left(\frac{\epsilon}{1-\delta}\right)^\alpha\right]\epsilon p_{n-1}^{1+\alpha}$$

or $(1 - \epsilon^\alpha) \geqslant 2(1 - \delta\epsilon^{1+\alpha})(1 - (\epsilon/(1 - \delta\epsilon))^\alpha)$. Some straightforward but tedious calculations show that no $\epsilon < \rho/(1 + \rho\delta)$ satisfies this inequality. *Q.E.D.*

Lemma 2. For every $\alpha > 0$ there exists a $\bar{\delta}(\alpha)$ such that for all δ satisfying $\bar{\delta}(\alpha) \leqslant \delta < 1$, the optimally collusive simple-strategy profile uses a rate of descent $\epsilon_+(\delta)$ defined by

$$\epsilon_+ \equiv \sup E, \qquad \text{where} \qquad E = \{\epsilon\colon \epsilon(1-\epsilon^\alpha) \geqslant 2\mu(1-\delta\epsilon)(1-\delta\epsilon^{1+\alpha})\},$$

[10] If $p_0 \geqslant \gamma^{-1}$, identical sales at identical prices occur under $(\epsilon p_0, \epsilon)$ as under (p_0, ϵ), but they occur one period sooner.

and sets an initial price,

$$p_0 = \frac{\rho(1-\delta)(1-\delta\epsilon_+^{1+\alpha})}{(1-\delta\epsilon_+)^{1+1/\alpha}}.$$

Proof. We have already argued that the optimal simple-strategy profile must satisfy $p_0 < \gamma^{-1}$. For any ϵ such simple-strategy profiles yield joint profits of

$$\Pi_0 = p_0 - \frac{(1-\delta\epsilon)^{1+\alpha}}{(1-\delta)^\alpha(1-\delta\epsilon^{1+\alpha})} p_0^{1+\alpha}.$$

The *unconstrained* maximum of this function occurs at

$$p_0 = \frac{\rho(1-\delta)(1-\delta\epsilon^{1+\alpha})}{(1-\delta\epsilon)^{1+1/\alpha}},$$

which exceeds $\rho(1-\delta)$ for all (ϵ, δ). Since the choice of p_0 does not affect subsequent incentive constraints (i.e., (i) in Lemma 1), the above equation defines the optimal p_0 as a function of ϵ. Substituting the optimal value for p_0 into the objective function yields joint profits of

$$\hat{\Pi}(\epsilon) = \mu\left(\frac{1-\delta}{1-\delta\epsilon}\right)\left(\frac{1-\delta\epsilon^{1+\alpha}}{1-\delta\epsilon}\right)^{1/\alpha}.$$

The *optimal* ϵ maximizes $\hat{\Pi}(\epsilon)$ subject to the incentive constraints (i). Since $\partial\hat{\Pi}/\partial\epsilon > 0$, finding the optimal ϵ amounts to choosing the largest value of ϵ that satisfies the incentive-compatibility constraints. We prove in Appendix A that the set E is nonempty if and only if $\delta \geqslant \bar{\delta}(\alpha)$, where $\bar{\delta}(\alpha) \leqslant .59$ for all $\alpha > 0$. *Q.E.D.*

Minimally collusive simple-strategy profiles (with $\Pi_0 > 0$) are established in Theorem A2 of Appendix A.

Next, we wish to show that the optimally collusive joint profits, denoted $\Pi_+(\delta)$ ($\equiv\hat{\Pi}(\epsilon_+(\delta))$), converge to the profits that a monopolist with commitment power could make. Recall from Stokey (1979) that this monopolist maximizes by charging the static monopoly price ρ forever. Consider any fixed, positive *real-time* rate of descent in price. If price follows the rule ρe^{-st}, where $s > 0$, and the interval between successive periods is z, we have $p_n = \rho e^{-nsz}$. Now let z approach zero. The equilibrium joint-profit stream, $\Pi_n = \sum_{i=0}^{\infty} \delta^i p_{n+i}(v_{n+i}^\alpha - v_{n+i+1}^\alpha)$, is negligibly affected by the choice of z. In particular, for all $n \geqslant 1$, Π_n converges to a positive constant times $v_n^{1+\alpha}$ as z approaches 0. But the profits Π_n^{dev} from optimally deviating equal $\mu(1-\delta)v_n^{1+\alpha}$, and $\mu(1-\delta)$ approaches 0 as z approaches 0. Hence, for small but positive s, there exists a largest z, call it $\bar{z}$, such that subgame perfection is satisfied in all but (possibly) the initial period whenever the time interval between periods does not exceed $\bar{z}$. If $\delta = e^{-r\bar{z}}$ and $\epsilon = e^{-s\bar{z}}$, then ϵ equals $\epsilon_+(\delta)$ of Lemma 2.

By setting s near zero, Stokey's (constant) intertemporal pricing scheme can be arbitrarily closely approximated. Incentive compatibility is preserved (in all periods $n \geqslant 1$), provided z is sufficiently small. Then, virtually all consumers with valuations exceeding ρ will purchase at price ρ in period zero, and this yields joint profits $\Pi_0 \approx \mu$ and guarantees incentive compatibility in period zero. We conclude that the optimally collusive joint payoff $\Pi_+(\delta)$ converges to static monopoly profits μ as δ approaches 1, i.e., as z approaches 0. Furthermore, the minimally collusive joint payoff, $\pi_-(\delta)$, converges to zero.

Lemma 3. $\epsilon_+(\delta)$ approaches 1 as δ approaches 1. Furthermore, $p_0(\epsilon_+)$ approaches ρ, $\Pi_+(\delta)$ approaches μ, and $\Pi_-(\delta)$ approaches 0.

Proof. $\epsilon_+(\delta)$ satisfies the equation,

$$\psi(\epsilon) = 2\mu\delta^2\epsilon^{2+\alpha} + (1-2\mu\delta)\epsilon^{1+\alpha} - (1+2\mu\delta)\epsilon + 2\mu = 0.$$

Appendix A establishes that $\epsilon_+(\delta)$ approaches 1 as δ approaches 1 and that $\lim_{\delta \to 1} \partial\epsilon_+/\partial\delta = 0$. Using l'Hôpital's rule, it is then easily verified that $p_0(\epsilon_+(\delta))$ approaches ρ and that $\hat{\Pi}(\epsilon_+(\delta))$ approaches μ. Also observe that the price path corresponding to $v_n = \epsilon_+^n$ is feasible and yields joint profits of $2\mu(1 - \delta)$. Thus, $\Pi_-(\delta) \leqslant 2\mu(1 - \delta)$ and $\lim_{\delta \to 1} \Pi_-(\delta) = 0$. *Q.E.D.*

4. The set of equilibrium joint profits

■ For every distribution function F, real interest rate $r > 0$, and time interval between periods $z > 0$, let $SPE(F, r, z)$ denote the *set of equilibrium joint profits,* i.e., the set of all $\Pi_0 = \pi_0^1 + \pi_0^2$ arising from subgame-perfect equilibria. We prove the following theorem.

Theorem 3. For every $\alpha > 0$, consider the durable goods oligopoly model in which consumers' valuations v are distributed by $F(v) = v^\alpha$ and consumers are allocated by an equal division rule when firms charge identical prices. Then there exist positive valued functions $\bar{\delta}(\alpha)$, $\Pi_+(\alpha, \delta)$, and $\Pi_-(\alpha, \delta)$ such that:

$$SPE(v^\alpha, r, z) = \begin{cases} \{0\} \cup [\Pi_-(\alpha, \delta), \Pi_+(\alpha, \delta)] & \text{if} \quad \delta \equiv e^{-rz} \geqslant \bar{\delta}(\alpha), \\ \{0\} & \text{if} \quad \delta \equiv e^{-rz} < \bar{\delta}(\alpha). \end{cases}$$

$\Pi_+(\alpha, \delta)$ was explicitly developed for duopoly in Section 3, and $\Pi_-(\alpha, \delta)$ and $\bar{\delta}(\alpha)$, which were defined in Section 3, are developed in Appendix A. Furthermore, $\lim_{\delta \to 1} \Pi_+(\delta) = \mu$ and $\lim_{\delta \to 1} \Pi_-(\delta) = 0$.

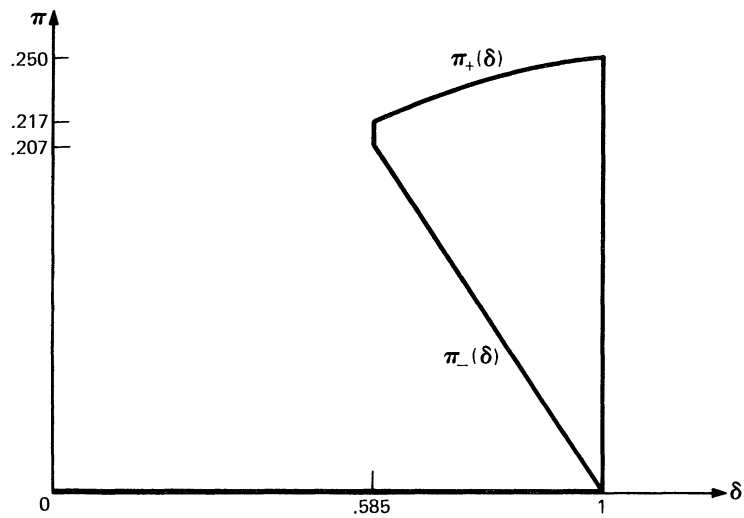
Proof. Suppose that $\delta \geqslant \bar{\delta}(\alpha)$. By Lemma 2, Theorem A1, and Theorem A2, optimally collusive and minimally collusive joint payoffs are supported by simple-strategy profiles (p_0, ϵ), with $\epsilon = \epsilon_+$ in each. Now observe that Π_0 is a continuous function in p_0 for fixed ϵ. Furthermore, the set of all p_0 satisfying the conditions of Lemma 1, for fixed ϵ, is connected. By continuously varying the initial price, we can obtain all intermediate payoffs while preserving subgame perfection. The Bertrand equilibrium of $p_n = 0$ for all n is also an equilibrium.

If $\delta < \bar{\delta}(\alpha)$, subgame perfection requires that $p_0 = 0$. The limiting result follows from Lemma 3. *Q.E.D.*

In Figure 1 we depict the set of equilibrium joint profits when consumer valuations

FIGURE 1

THE SET OF EQUILIBRIUM JOINT PROFITS, WITH F(v) = v, FOR ALL DISCOUNT FACTORS δ



are uniformly distributed over [0, 1] (i.e., $\alpha = 1$). When the discount factor δ is less than $\bar{\delta}(1) \approx .585$, the inequality $\epsilon(1 - \epsilon) \geqslant .5(1 - \delta\epsilon)(1 - \delta\epsilon^2)$ has no solutions for ϵ in [0, 1], and the Bertrand equilibrium is the unique subgame-perfect equilibrium. At $\delta = \bar{\delta}$ the unique rate of descent $\epsilon = .7235$ is viable for positive p_0, but p_0 may range anywhere from .345 to .522, and thereby yield joint profits anywhere from .207 to .217. Then, as δ increases to one, the set of equilibrium joint profits spreads to the entire interval from zero to static monopoly profits.

The behavior *away from the limit* implied by Theorem 3 (and Figure 1) is striking. First, observe that for $\bar{\delta}(\alpha) \leqslant \delta < 1$, the equilibrium set is *not* connected. Marginal-cost pricing is always an equilibrium, but to support tacit collusion, profits must be bounded away from zero. The intuition for this result is that future profits are the inducement to prevent present deviations. If future profits, compared with the highest remaining consumer valuation, are excessively low, it necessarily pays for one firm to undercut its rival. Second, observe that effective collusion is quite possible far away from the limit. Even when $\delta = .585$, the duopolists, if they collude at all, must earn at least 83% of the static monopoly surplus, and possibly as much as 87%.

The effect of varying the time interval z between successive offers on optimal collusion by oligopolists is exactly the *opposite* of the effect such variation has on a durable goods monopolist confined by the Coase conjecture. When z approaches infinity (i.e., $\delta \to 0$), the durable goods monopolist becomes, for all practical purposes, a static monopolist and hence can approach static monopoly profits. Meanwhile, the oligopolists find themselves playing, for all practical purposes, a one-shot Bertrand game. On the other hand, as z approaches zero (i.e., $\delta \to 1$), the monopolist in a Coase conjecture[11] subgame-perfect equilibrium drops to marginal cost "in the twinkling of an eye" (Coase, 1972). When there are at least two firms in the market, however, the ability to collude increases as z goes to zero.

The intuition for the oligopoly result is clear: the incentive to cheat on any collusive agreement (enforced by the type of trigger strategies discussed above) goes to zero as z approaches zero, whereas the loss from future retaliation stays roughly constant. The incentive to cheat vanishes because of the anticipatory behavior of consumers. As z becomes smaller, fewer and fewer consumers will be willing to buy from the price cutter because they expect a better deal (a price equal to marginal cost) in the next round.

This intuition makes it clear that the limiting result of Theorem 3 does not depend on our assumption that $F(v) = v^\alpha$. Indeed, the main theorem in Gul (1987) is stated for arbitrary demand curves. Furthermore, it is easy to argue that the behavior away from the limit (depicted in Figure 1) holds quite generally. For arbitrary $F(\cdot)$, define $\underline{v}$ and $\bar{v}$ by $F(\underline{v}) = 0$, $F(\bar{v}) = 1$, and $0 < F(v) < 1$ whenever $\underline{v} < v < \bar{v}$. Now suppose there exist $\alpha > 0$ and $L \geqslant M > 0$ such that

$$M(v-\underline{v})^\alpha \leqslant F(v) \leqslant L(v-\underline{v})^\alpha, \qquad \text{for all } v \text{ such that} \qquad \underline{v} \leqslant v \leqslant \bar{v}, \tag{3}$$

i.e., $F(v)$ is "enveloped" by $L(v - \underline{v})^\alpha$ and $M(v - \underline{v})^\alpha$. In Appendix B we demonstrate that when (3) is satisfied,[12] tight bounds showing rapid convergence may also be established. Broadly speaking, large profits are possible even when the time interval between periods is very long.

[11] But also see the Conclusion—and Ausubel and Deneckere (1986)—, which discuss the class of *all* subgame-perfect equilibria for monopoly for the case in which $\underline{v} = 0$, where $F(\underline{v}) = 0$ and $F(v) > 0$ for all $v > \underline{v}$.

[12] Suppose, for example, that F is differentiable in a neighborhood of $\underline{v}$, and that there exist $l \geqslant m > 0$ such that $m \leqslant F'(v) \leqslant l$ in that neighborhood. Then condition (3) may be shown to be satisfied by using $\alpha = 1$. Even if $F'(\underline{v}) = 0$ or $F'(\underline{v}) = \infty$, it is still often possible to satisfy (3) by using some $\alpha \neq 1$.

5. Potential entry and the Coase conjecture

■ The contestable markets literature modifies the theory of static monopoly by arguing that the existence of a *potential entrant* may force a monopolist to price at marginal cost. The Coase conjecture literature further amends our understanding of monopoly by observing that a monopolist in a durable good may be forced, owing to lack of commitment power and anticipatory behavior on the part of consumers, to price at close to marginal cost. We now combine these forces and demonstrate that the existence of a potential entrant may enable a durable goods monopolist to price close to the static monopoly price. In other words, one (and one potential entrant) is enough for monopoly.

Let us make precise the sequencing of moves. At the beginning of every period, the entrant has the choice of whether to enter if he has not entered previously. If the entrant decides to stay out, the incumbent names a price (after observing the entrant's decision), and consumers make their purchasing decisions. The game then repeats in the same fashion in the next period. If the potential entrant decides to join the market, both firms simultaneously and independently call out prices (after observing the entry decision), which the consumers can then either accept or reject.[13] The play proceeds in subsequent periods with the two firms' naming prices followed by the consumers' making purchases.

To make the description of the equilibrium we have in mind compact, it is convenient to consider three types of outcome paths. The equilibrium will incorporate these three paths, both in specifying an initial outcome path and in determining punishments for any deviation from the initial outcome path or from ongoing punishments.

Path 1. The incumbent charges close to the static monopoly price; the potential entrant does not enter.

This is the equilibrium path. Let $p_0(\epsilon)$ denote the initial price of the optimally collusive duopoly solution and s the real-time rate of descent of prices. The incumbent charges a price $p_n = \epsilon^n p_0(\epsilon)$ in each period n, where $\epsilon = e^{-sz}$. Meanwhile, the potential entrant stays out. Thus, along path 1, the incumbent earns the joint duopoly profits associated with the simple-strategy profile (p_0, ϵ).

Path 2. The potential entrant enters; the two firms then follow a subgame-perfect equilibrium price path.

This is the punishment path if the incumbent deviates from path 1. The punishment path is characterized by two numbers: (ω, ϵ). The interpretation of ω and ϵ is the following: if x is the price at which a deviation from path 1 occurred, and if an offer of x induces all customers with valuations $\geqslant v$ to buy, both entrant and incumbent charge a price of ωv in the next period. Afterwards, the price is discounted by the fraction $\epsilon = e^{-sz}$ every period. The punishment path after such a deviation is thus $(\omega v, \omega\epsilon v, \omega\epsilon^2 v, \omega\epsilon^3 v, \ldots)$.

Path 3. The incumbent and the entrant revert to marginal-cost pricing.

This is the punishment path if the entrant deviates from path 1 or if either firm deviates from path 2.

[13] More true to the spirit of the contestable markets literature, one could allow the entrant a first-mover advantage by not permitting the incumbent to react until the period following entry. Uninvited entry will yield the entrant only limited profits, since consumers expect an all-out price war in subsequent periods. Small sunk entry costs proportional to the size of the market remaining (representing, e.g., introductory advertising costs) still make the entrant prefer to stay out when uninvited, but wanting to enter when invited. The monopolist's profits, however, will be bounded away from static monopoly profits by an amount equal to sunk entry costs.

We thank Vijay Krishna for raising this issue.

The equilibrium goes as follows. The incumbent follows path 1. If the incumbent ever deviates (singly) from path 1, he triggers a reversion to path 2. If either the incumbent or the entrant deviates from path 2, or if the entrant ever deviates from path 1 (singly or jointly with the incumbent), both players revert to path 3. Deviations from path 3 are punished by starting that path over again. Observe that under the above strategies, the potential entrant has a strict incentive to enter whenever he is supposed to, but may as well stay out as long as the monopolist follows path 1.

Before showing that the strategies described form an equilibrium strategy pair for large enough δ, we offer one possible interpretation of the equilibrium. As long as the monopolist sticks to path 1, the potential entrant stays out of the market, because he interprets this behavior as a sign of determination on the part of the monopolist. "Mean and nasty" monopolists not only take advantage of consumers but also retaliate against new entrants. Deviations from path 1 are considered to be a sign of weakness on the part of the monopolist, and they lead to the entrant's inference that the monopolist will not spoil the market once entry occurs (because weak monopolists are soft on entrants as well as consumers). Such behavior invites entry.

To establish that the strategies above induce an equilibrium for large enough δ, we need to show the existence of a value of ω such that path 2 is a subgame-perfect equilibrium starting from any deviation x and such that the monopolist has no incentive to trigger a reversion to that path.[14] We prove these two results in Appendix B. Observe that as the real-time rate of descent s approaches zero, the monopolist's initial price converges to ρ with corresponding profits of μ. Hence we obtain the following theorem.

Theorem 4. For every $r > 0$ and $\theta > 0$, there exists a $\bar{z} > 0$ such that for every z satisfying $0 < z < \bar{z}$, the incumbent firm in the potential entry model earns profits greater than $\mu - \theta$ in the above-described equilibrium.

6. Conclusion

■ In this article we have proved that a durable goods monopolist may benefit from entry or potential entry. Whereas the monopolist lacks the means to force his future self to follow a strategy his present self would like, the oligopolist finds commitment power in the actions of his rivals. To put it bluntly: "If you cannot punish yourself, find someone else to punish you." We shall conclude by comparing our model with related literature in which punishment enables improved payoffs and by relating our results obtained here to the literature on durable goods monopoly.

Traditional supergame analysis of the oligopoly problem (Abreu, 1985) also makes extensive use of punishment strategies to support collusive outcomes. It is instructive to compare the modelling in that literature with the present article's. As a supergame is typically an infinitely repeated version of a one-shot game, the supergame analysis of oligopoly assumes that firms face the same static demand curve in every period. In contrast, our model contains a convenient (albeit extreme) version of *intertemporal substitutability in demand.* Our results indicate that intertemporal substitutability in demand facilitates collusion, via rational expectations on the part of consumers. Suppose consumers witness a deviation from cooperative behavior by one firm. Given the subgame-perfect equilibrium (supported by punishment strategies), consumers anticipate an all-out price war in subsequent periods and postpone their purchases. *Ceteris paribus,* a firm's one-period gain associated with deviation will be less than it is in the standard supergame treatment of oligopoly, and this enables more collusive outcomes to be supported as subgame-perfect equilibria. Moreover, the effect we

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[14] Consumer expectations are fixed (in a fashion similar to the previous section) to implement this equilibrium. See footnote 7.

are describing appears to be a real phenomenon: for example, a consumer who sees one airline cut its fare can often profit by deferring purchase of his ticket, as price-matching by other airlines and further cuts may reasonably be expected.

Closely related to the Coase conjecture is the literature on the time-consistency problem of macroeconomics (Kydland and Prescott, 1977). In a variety of contexts the government may seek to choose a sequence of policy actions over time that are not "time-consistent": if the government were able to reoptimize in subsequent periods, it would not choose the policy actions called for under the original maximization. Hence, unless the government possesses commitment power, it cannot follow such a strategy. Our approach suggests a way out of this conundrum: explicit division of authority may permit mutually assured commitment. For example, if we give Congress the authority to run deficits but only give an independent Federal Reserve Board the power to monetize the debt, we may permit a subgame-perfect equilibrium where the debt is never monetized.

Finally, we can compare the models of oligopoly and potential entry in this article with the model of pure durable goods monopoly[15] in a sequel (Ausubel and Deneckere, 1986). Two cases need to be distinguished. First, consider the case of a "gap" between seller's cost and buyers' valuations. When $\underline{v} > 0$, it has been shown (Fudenberg, Levine, and Tirole, 1985; Gul, Sonnenschein, and Wilson, 1986) that there generically exists a unique subgame-perfect equilibrium in the pure monopoly model. Moreover, as the time interval z approaches zero, the initial price p_0 necessarily approaches $\underline{v}$, which may be much lower than the static monopoly price. In contrast, for oligopoly and monopoly with potential entry, a folk theorem holds. The explanation for the oligopolists' advantage lies in the fact that, when $\underline{v} > 0$ and $\delta < 1$, all sales occur in *finite* time in any monopoly subgame-perfect equilibrium. Backward induction from the last period of positive sales drives the initial price near $\underline{v}$. But with entry or potential entry, a pricing rule similar to $p_n - \underline{v} = \epsilon^n(p_0 - \underline{v})$ becomes incentive-compatible. Sales are then extended over an infinite time, and backward induction fails.

Second, consider the case where there is "no gap" between the seller's marginal cost and the lowest buyer valuation. We demonstrate in Ausubel and Deneckere (1986) that when $\underline{v} = 0$, and under very general distributional assumptions, there exist subgame-perfect equilibria in which the monopolist initially charges the static monopoly price and then follows a very slow rate of price descent. This main equilibrium path is supported by a reversion to a Coase conjecture subgame-perfect equilibrium if the monopolist ever deviates. Hence, in the case of no gap, a durable goods monopolist may earn monopoly profits. Nevertheless, observe (somewhat counterintuitively) that for a wide range of discount factors, the duopolists may do strictly better than the monopolist. Reversion to the Bertrand equilibrium is more severe than reversion to a Coase conjecture subgame-perfect equilibrium so that the optimally collusive duopoly equilibrium yields higher joint profits than the maximally profitable monopoly subgame-perfect equilibrium.

Combining the results of the two articles, we may conclude that in the case of no gap between seller's cost and buyers' valuations, one firm is enough for a durable goods industry to earn monopoly profits (though more than one may do better). In the case of a gap, one is not enough for monopoly, but one is almost enough.

Appendix A

■ We derive conditions on the subgame-perfect equilibria of a durable goods duopoly with an "equal division" rule: optimally collusive payoffs and minimally collusive payoffs are proved to be supported by simple-strategy profiles for all $\alpha > 0$. We can derive analogous results for an oligopoly with N firms.

[15] It has been observed by a number of authors that the durable goods monopoly model is formally equivalent to a sequential bargaining model of one-sided offers and one-sided uncertainty. Thus, this article has implications for the multilateral bargaining problem with two sellers (of known valuation) and one buyer (of unknown valuation).

We first write equations that express $\{p_n\}_{n=0}^{\infty}$ and $\{\Pi_n\}_{n=0}^{\infty}$ in terms of $\{v_n\}_{n=0}^{\infty}$. If the consumer with valuation v_{n+1} is indifferent between purchasing in periods n and $n+1$, $v_{n+1} - p_n = \delta(v_{n+1} - p_{n+1})$, giving the difference equation $p_n = (1-\delta)v_{n+1} + \delta p_{n+1}$. Telescoping the right side yields:

$$p_n = (1-\delta)\sum_{i=0}^{\infty}\delta^i v_{n+i+1}, \qquad \text{for all} \qquad n \geq 0. \tag{A1}$$

Equation (A1) was developed in Stokey (1981). Meanwhile, the net present value of joint profits, Π_n, is defined by

$$\Pi_n = \sum_{i=0}^{\infty}\delta^i p_{n+i}(v_{n+i}^{\alpha} - v_{n+i+1}^{\alpha}), \qquad \text{for all} \qquad n \geq 0. \tag{A2}$$

Conditions (1) and (2) of Theorem 2 hold for all subgame-perfect equilibria. The following lemma shows that (2) is vacuous when $\alpha \geq 1$.

Lemma A1. Consider any subgame-perfect equilibrium such that $p_n > 0$ for all $n \geq 0$, and let $\alpha \geq 1$. Then $p_n \geq \rho(1-\delta)v_n$ for all $n \geq 0$.

Proof. Suppose not. Then there exists n such that $0 < p_n < \rho(1-\delta)v_n$. Using inequality (2) of Theorem 2, observe that $\Pi_n = \pi_n^1 + \pi_n^2 \geq [2\alpha/(1+\alpha)]p_n v_n^{\alpha}$. Note, however, that joint profits along the equilibrium path are bounded above by the current price times the number of remaining customers. Hence, $\Pi_n < p_n v_n^{\alpha}$, which gives a contradiction when $\alpha/(1+\alpha) \geq 1/2$. *Q.E.D.*

Lemma A1 is not true for $\alpha < 1$. We can still show that (2) is irrelevant for optimally collusive subgame-perfect equilibria.

Lemma A2. Suppose that $\{v_n\}_{n=0}^{\infty}$ maximizes Π_0 subject to:

(a) $\Pi_n \geq 2\mu(1-\delta)v_n^{1+\alpha}$, for all $n \geq 1$ such that $p_n \geq \rho(1-\delta)v_n$,

(b) $\Pi_n \geq 2p_n\left(v_n^{\alpha} - \left(\dfrac{p_n}{1-\delta}\right)^{\alpha}\right)$, for all $n \geq 1$ such that $p_n < \rho(1-\delta)v_n$, and

(c) $v_0 = 1$, and $v_n \geq v_{n+1} \geq 0$, for all $n \geq 0$,

where $\{p_n\}_{n=0}^{\infty}$ is defined by (A1) and $\{\Pi_n\}_{n=0}^{\infty}$ is defined by (A2).

Then for any $\alpha > 0$, if $p_n > 0$ (for all $n \geq 0$), we have:

$$\Pi_n \geq 2\mu(1-\delta)v_n^{1+\alpha}, \qquad \text{for all} \qquad n \geq 1.$$

Outline of proof. Suppose not. Let $k = \inf\{n \geq 1: \Pi_n < 2\mu(1-\delta)v_n^{1+\alpha}\}$. Observe that $v_k > v_{k+1}$. For β satisfying $0 < \beta < v_k/v_{k+1}$, define $\{v'_n\}_{n=0}^{\infty}$ by equation (A4) (in the proof of the next lemma). $\{p'_n\}_{n=0}^{\infty}$ and $\{\Pi'_n\}_{n=0}^{\infty}$ are defined by (A1) and (A2). Define $a_n(\beta)$ to be the slack in the nth constraint (a), evaluated at $\{v'_n\}_{n=0}^{\infty}$ given by β, and define $b_n(\beta)$ to be the slack in the nth constraint (b). The following can be established by direct calculation:

$$\left.\frac{\partial \Pi'_0}{\partial \beta}\right|_{\beta=1} > 0;$$

$$\left.\frac{\partial a_n}{\partial \beta}\right|_{\beta=1} > 0, \qquad \text{and} \qquad \left.\frac{\partial b_n}{\partial \beta}\right|_{\beta=1} > 0, \qquad \text{for all} \qquad n \leq k-1;$$

$$\left.\frac{\partial b_k}{\partial \beta}\right|_{\beta=1} > 0; \qquad \text{and}$$

$$a_n(\beta) = \beta^{1+\alpha}a_n(1) \qquad \text{and} \qquad b_n(\beta) = \beta^{1+\alpha}b_n(1) \qquad \text{for all} \qquad n \geq k+1.$$

Hence, we can conclude that there exists β $(1 < \beta < v_k/v_{k+1})$ such that $\{v'_n\}_{n=0}^{\infty}$ is in the feasible region of the above maximization problem, but $\Pi'_0 > \Pi_0$, which contradicts our hypothesis that $\{v_n\}_{n=0}^{\infty}$ maximizes Π_0. *Q.E.D.*

Using Theorem 2, Lemma A1, and Lemma A2, we show that the optimally collusive subgame-perfect equilibrium must solve the following optimization problem for all $\alpha > 0$:

$$\max_{\{v_n\}_{n=0}^{\infty}} \Pi_0 \tag{A3}$$

subject to:

$$\Pi_n \geq 2\mu(1-\delta)v_n^{1+\alpha}, \qquad \text{for all} \qquad n \geq 1, \tag{A3a}$$

and

$$v_0 = 1 \qquad \text{and} \qquad v_n \geq v_{n+1} \geq 0, \qquad \text{for all} \qquad n \geq 0. \tag{A3b}$$

Lemma A3. For any $\alpha > 0$, suppose that $\{v_n\}_{n=0}^{\infty}$ solves (A3) and $\Pi_0 > 0$. Then constraint (A3a) holds with equality, i.e.,

$$\Pi_n = 2\mu(1-\delta)v_n^{1+\alpha}, \qquad \text{for all} \qquad n \geq 1. \tag{A4}$$

Proof. Suppose not. We shall demonstrate an alternative sales path $\{v'_n\}_{n=0}^{\infty}$ that improves upon Π_0, while still satisfying constraint (A3a).

Observe that $v_n > 0$ for all $n \geq 0$, by induction: $\Pi_n > 0$ implies that $p_n > 0$, and so $v_{n+1} > 0$, but by (A3a), $\Pi_{n+1} > 0$, etc.

Define k to be the first period in which constraint (A3a) has slack *and* in which sales are positive: $k = \inf\{n \geq 1\colon \Pi_n > 2\mu(1-\delta)v_n^{1+\alpha}$ and $v_n > v_{n+1}\}$. Note that $k < \infty$ if and only if the hypothesis of the lemma is not satisfied. Suppose there exists n such that $\Pi_n > 2\mu(1-\delta)v_n^{1+\alpha}$, but $v_n = v_{n+1}$. Since there are no sales in period n, $\Pi_n = \delta\Pi_{n+1}$, which implies that $\Pi_{n+1} > 2\mu(1-\delta)v_{n+1}^{1+\alpha}$. Eventually, we must reach $m > n$ such that $v_m > v_{m+1}$, since v_n is positive, but $\lim_{i\to\infty} v_i = 0$ (by Theorem 1).

Let β satisfy $0 < \beta < v_k/v_{k+1}$. Define $\{v'_n\}_{n=0}^{\infty}$ by

$$v'_n = \begin{cases} v_n, & \text{if} \quad n \leq k-1; \\ [p_{k-1} - \beta\delta p_k]/[1-\delta], & \text{if} \quad n = k; \\ \beta v_n, & \text{if} \quad n \geq k+1. \end{cases} \tag{A5}$$

Observe that $\{v'_n\}_{n=0}^{\infty}$ has been conveniently chosen so that by equation (A1) it determines $\{p'_n\}_{n=0}^{\infty}$ given by

$$p'_n = \begin{cases} p_n, & \text{if} \quad n \leq k-1; \\ \beta p_n, & \text{if} \quad n \geq k. \end{cases} \tag{A6}$$

It is also straightforward to verify from equation (A2) that

$$\Pi'_n = \beta^{1+\alpha}\Pi_n, \qquad \text{for all} \qquad n \geq k+1. \tag{A7}$$

Furthermore, for $n \leq k-2$,

$$\Pi'_n = [\sum_{i=0}^{k-2-n} \delta^i(v'^{\alpha}_{n+i} - v'^{\alpha}_{n+i+1})p'_{n+i}] + \delta^{k-1-n}\Pi'_{k-1}$$

and

$$\Pi_n = [\sum_{i=0}^{k-2-n} \delta^i(v^{\alpha}_{n+i} - v^{\alpha}_{n+i+1})p_{n+i}] + \delta^{k-1-n}\Pi_{k-1},$$

which implies that

$$\Pi'_n = \Pi_n + \delta^{k-1-n}[\Pi'_{k-1} - \Pi_{k-1}], \qquad \text{for all} \qquad n \leq k-2. \tag{A8}$$

Claim. $\partial\Pi'_{k-1}/\partial\beta$, evaluated at $\beta = 1$, is strictly positive.

Proof of claim. Joint profits, Π'_{k-1}, may be written as

$$\Pi'_{k-1} = (v'^{\alpha}_{k-1} - v'^{\alpha}_{k})p'_{k-1} + \delta(v'^{\alpha}_{k} - v'^{\alpha}_{k+1})p'_k + \delta^2\Pi'_{k+1}.$$

Its derivative with respect to β is calculated by using (A5), (A6), and (A7):

$$\begin{aligned} \frac{\partial\Pi'_{k-1}}{\partial\beta} &= \left[\frac{\partial v'^{\alpha}_{k-1}}{\partial\beta} - \frac{\partial v'^{\alpha}_{k}}{\partial\beta}\right]p'_{k-1} + \delta\left[\frac{\partial v'^{\alpha}_{k}}{\partial\beta} - \frac{\partial v'^{\alpha}_{k+1}}{\partial\beta}\right]p'_k + \delta[v'^{\alpha}_k - v'^{\alpha}_{k+1}]\frac{\partial p'_k}{\partial\beta} + \delta^2\frac{\partial\Pi'_{k+1}}{\partial\beta} \\ &= \alpha\delta p_k v'^{\alpha-1}_k\left\{\frac{p'_{k-1} - \delta p'_k}{1-\delta}\right\} - (1+\alpha)\delta p_k v'^{\alpha}_{k+1} + \delta p_k v'^{\alpha}_k + (1+\alpha)\delta^2\beta^{\alpha}\Pi_{k+1}. \end{aligned}$$

The above expression in braces equals v'_k. Moreover, at $\beta = 1$, we have $v'_k = v_k$ and $v'_{k+1} = v_{k+1}$. Hence,

$$\left.\frac{\partial\Pi'_{k-1}}{\partial\beta}\right|_{\beta=1} = (1+\alpha)[\delta p_k(v^{\alpha}_k - v^{\alpha}_{k+1}) + \delta^2\Pi_{k+1}] > 0,$$

which proves the claim.

Remainder of the Proof of Lemma A3. By hypothesis, constraint (A3a) has slack for $n = k$, and sales are positive in period k. With the claim and (A5), there exists β satisfying $1 < \beta < v_k/v_{k+1}$ such that

$$\Pi'_k > 2\mu(1-\delta)v'^{1+\alpha}_k \qquad \textit{and} \qquad \Pi'_{k-1} > \Pi_{k-1}. \tag{A9}$$

Recall that $\{v_n\}_{n=0}^{\infty}$ satisfies constraint (A3a). By equations (A5) and (A7),

$$\Pi'_n = \beta^{1+\alpha}\Pi_n \geq 2\mu(1-\delta)\beta^{1+\alpha}v_n^{1+\alpha} = 2\mu(1-\delta)v'^{1+\alpha}_n, \qquad \text{for all} \qquad n \geq k+1,$$

and by equations (A5), (A8) and (A9),

$$\Pi_n' > \Pi_n \geq 2\mu(1-\delta)v_n^{1+\alpha} = 2\mu(1-\delta)v_n'^{1+\alpha}, \qquad \text{for all} \qquad n \leq k-1.$$

Thus, $\{v_n'\}_{n=0}^{\infty}$ satisfies constraint (A3a) and $\Pi_0' > \Pi_0$. We conclude that $\{v_n\}_{n=0}^{\infty}$ is *not* a solution to (A3), which contradicts the hypothesis. *Q.E.D.*

Next, we use Lemma A3 to derive a (nonlinear) difference equation that the solution to (A3) must satisfy. Observe that

$$\Pi_n = p_n(v_n^\alpha - v_{n+1}^\alpha) + \delta\Pi_{n+1}, \qquad \text{for all} \qquad n \geq 0, \tag{A10}$$

$$p_n - \delta p_{n+1} = (1-\delta)v_{n+1}, \qquad \text{for all} \qquad n \geq 0. \tag{A11}$$

The solution to (A3) satisfies equation (A4). Substituting (A4) into (A10) gives

$$p_n = \frac{\Pi_n - \delta\Pi_{n+1}}{v_n^\alpha - v_{n+1}^\alpha} = 2\mu(1-\delta)\left(\frac{v_n^{1+\alpha} - \delta v_{n+1}^{1+\alpha}}{v_n^\alpha - v_{n+1}^\alpha}\right). \tag{A12}$$

Then, substituting (A12) into (A11) yields

$$\frac{v_n^{1+\alpha} - \delta v_{n+1}^{1+\alpha}}{v_n^\alpha - v_{n+1}^\alpha} - \delta\left(\frac{v_{n+1}^{1+\alpha} - \delta v_{n+2}^{1+\alpha}}{v_{n+1}^\alpha - v_{n+2}^\alpha}\right) = \frac{v_{n+1}}{2\mu}.$$

Let $\epsilon_n = v_{n+1}/v_n$. Then this equation may be rewritten as

$$\epsilon_n^{-1}\left(\frac{1-\delta\epsilon_n^{1+\alpha}}{1-\epsilon_n^\alpha}\right) - \delta\left(\frac{1-\delta\epsilon_{n+1}^{1+\alpha}}{1-\epsilon_{n+1}^\alpha}\right) = \frac{1}{2\mu}.$$

Setting $h_\alpha(z) = (1 - \delta z^{1+\alpha})/(1 - z^\alpha)$, we obtain our difference equation:

$$\epsilon_n^{-1}h_\alpha(\epsilon_n) - \delta h_\alpha(\epsilon_{n+1}) = \frac{1}{2\mu}, \qquad \text{for all} \qquad n \geq 1. \tag{A13}$$

We may establish the following properties of $h_\alpha(z)$: $\lim_{z\to 0} h_\alpha(z) = h_\alpha(0) = 1$; $\lim_{z\to 1} h_\alpha(z) = +\infty$; and $h_\alpha(\cdot)$ is monotonically *increasing* on [0, 1]. Let us rewrite (A13) as

$$h_\alpha(\epsilon_{n+1}) = \frac{1}{\delta}\left[\epsilon_n^{-1}h_\alpha(\epsilon_n) - \frac{1}{2\mu}\right]. \tag{A14}$$

The above properties show that a solution $\epsilon_{n+1} = g(\epsilon_n)$ to (A14) exists for ϵ_n close to 0 and 1, and that $\lim_{\epsilon_n\to 0} g(\epsilon_n) = \lim_{\epsilon_n\to 1} g(\epsilon_n) = 1$.

Furthermore, one easily establishes that the set of ϵ_n for which no solution $0 \leq \epsilon_{n+1} \leq 1$ to (A14) exists, i.e., the set of all ϵ_n such that $\epsilon_n^{-1}h_\alpha(\epsilon_n) - (1/2\mu) < \delta$, is an open interval. When no solution $\epsilon_{n+1} \leq 1$ to (A14) exists, we extend the definition of g to obtain a continuous function by setting $\epsilon_{n+1} = 0$. Some additional algebra shows that $g(\cdot)$ is first decreasing and then increasing, so we obtain the picture in Figure A1.

As drawn in Figure A1, $g(\cdot)$ has a flat section and exactly two fixed points in [0, 1). But it is also possible that $g(\epsilon)$ contains no flat sections ($g > 0$ on [0, 1]) or that g has no fixed points (the graph of g lies completely above the 45° line in [0, 1)). We now establish the existence of a $\bar{\delta}(\alpha)$ such that g has exactly two fixed points in [0, 1) for $\delta > \bar{\delta}(\alpha)$ and no fixed points if $\delta < \bar{\delta}(\alpha)$. Fixed points of $g(\cdot)$ are zeros of $\psi(\epsilon; \alpha, \delta)$, where

$$\psi(\epsilon; \alpha, \delta) = 2\mu\delta^2\epsilon^{2+\alpha} + (1 - 2\mu\delta)\epsilon^{1+\alpha} - (1 + 2\mu\delta)\epsilon + 2\mu,$$

ψ is a quasiconcave function, decreasing on $[0, \rho]$ and convex on $[\rho, 1]$, with $\psi(0) = 2\mu > 0$ and $\psi(1) = 2\mu(1 - \delta)^2 > 0$. Thus, either ψ has exactly two roots or no roots at all, except at some critical $\bar{\delta}(\alpha)$. Now, for each α, $\psi(1; \alpha, 1) = 0$, $\partial\psi/\partial\delta = 2\mu\epsilon[\epsilon^\alpha(2\delta\epsilon - 1) - 1)] < 0$, and $\partial\psi(1; \alpha, 1)/\partial\epsilon = \alpha > 0$. An application of the implicit function theorem thus yields that there exists a root in [0, 1) to $\psi(\epsilon; \alpha, \delta) = 0$ for δ close to 1. Since $\partial\psi/\partial\delta < 0$, we see that there exists $\bar{\delta}(\alpha)$ such that ψ has roots for $\delta \geq \bar{\delta}(\alpha)$. Some numerical calculations yield the graph of $\bar{\delta}(\alpha)$ versus $\alpha/(1 + \alpha)$ in Figure A2.

For $\bar{\delta}(\alpha) < \delta < 1$, let ϵ_- and ϵ_+ denote the smallest and largest fixed point of g, respectively, in [0, 1). Construct a square with vertices (ϵ_+, ϵ_+) and $(\hat{\epsilon}, \hat{\epsilon})$, where $\hat{\epsilon}$ is the unique $\epsilon < \epsilon_+$ such that $g(\hat{\epsilon}) = \epsilon_+$.

Consider $g^n(\epsilon)$. If $\epsilon \in [0, \hat{\epsilon})$ or $\epsilon \in (\epsilon_+, 1]$, then $g^n(\epsilon) \uparrow 1$. If $\epsilon \in [\hat{\epsilon}, \epsilon_+]$, then one of three possibilities holds:

$$g^n(\epsilon) > 0 \qquad \text{for all } n \text{ but} \quad g^k(\epsilon) \not\in [\hat{\epsilon}, \epsilon_+] \qquad \text{for some } k; \tag{A15}$$

$$g^n(\epsilon) = 0 \qquad \text{for some } n; \qquad \text{or} \tag{A16}$$

$$\hat{\epsilon} \leq g^n(\epsilon) \leq \epsilon_+ \qquad \text{for all } n. \tag{A17}$$

FIGURE A1

GRAPH OF g(•), THE SOLUTION TO THE DIFFERENCE EQUATION (A 14)

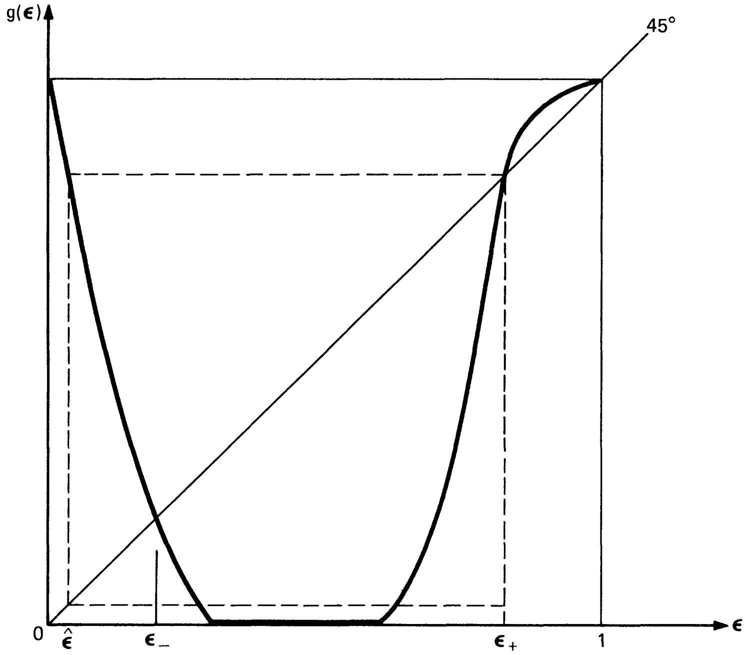
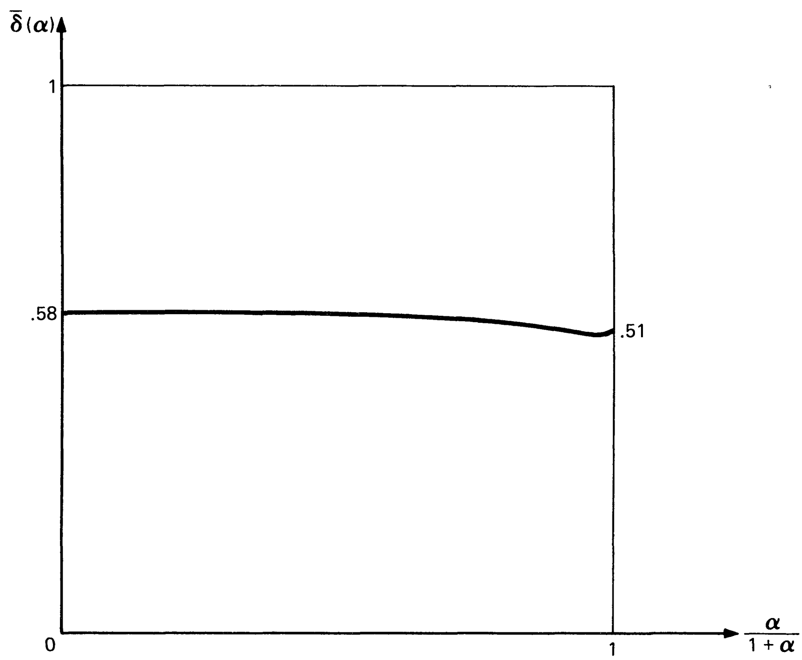


FIGURE A2

GRAPH OF $\bar{\delta}(\alpha)$, THE CRITICAL VALUE AT WHICH ROOTS OF ψ APPEAR



Lemma A4. For any $\alpha > 0$, suppose that $\{v_n\}_{n=0}^{\infty}$ solves (A3) and $\Pi_0 > 0$. Then:

$$\hat{\epsilon} \leqslant \frac{v_{n+1}}{v_n} \leqslant \epsilon_+, \qquad \text{for all} \qquad n \geqslant 1. \tag{A18}$$

Proof. Suppose that $v_2/v_1 < \hat{\epsilon}$ or $v_2/v_1 > \epsilon_+$. We have just argued that $\lim_{n\to\infty} (v_{n+1}/v_n) = 1$. For any $c < 1$, there exists n such that for every $i \geqslant n$, $v_{i+1} > cv_i$. Then

$$v_i^{\alpha} - v_{i+1}^{\alpha} < (1 - c^{\alpha})v_i^{\alpha} \leqslant (1 - c^{\alpha})v_n^{\alpha}, \qquad \text{for all} \qquad i \geqslant n.$$

Meanwhile, $p_i \leqslant v_n$ for all $i \geqslant n$, so by (A2):

$$\Pi_n < \sum_{i=1}^{\infty} \delta^i (1 - c^{\alpha}) v_n^{1+\alpha} = \left(\frac{1 - c^{\alpha}}{1 - \delta}\right) v_n^{1+\alpha}. \tag{A19}$$

But c may be chosen arbitrarily close to 1, and then (A19) contradicts (A4) (or simply (A3a)).

Suppose that $\hat{\epsilon} \leqslant v_2/v_1 \leqslant \epsilon_+$. We have just argued that one of (A15), (A16), or (A17) holds. If (A15) holds, then $v_{k+2}/v_{k+1} < \hat{\epsilon}$ or $v_{k+2}/v_{k+1} > \epsilon_+$, and we generate the contradiction of the previous paragraph. If (A16) holds, let $k = \inf\{n: g^n(\epsilon) = 0\}$. Then $v_{k+2}/v_{k+1} \leqslant 0$, which implies that $v_{k+2} \leqslant 0$, but $v_{k+1} > 0$. Note that $v_{k+2} \leqslant 0$ implies that $p_{k+1} \leqslant 0$. But then $\Pi_{k+1} \leqslant 0$, which contradicts (A4) (or simply (A3a)). The only remaining possibility is (A17). *Q.E.D.*

Theorem A1. For any $\alpha > 0$, suppose that $\{v_n\}_{n=0}^{\infty}$ solves (A3) and $\Pi_0 > 0$. Then $\{v_n\}_{n=0}^{\infty}$ corresponds to a simple-strategy profile, i.e., $v_{n+1}/v_n = \epsilon_+$ for all $n \geqslant 1$.

Proof. By Lemma A4, $\{v_n\}_{n=0}^{\infty}$ satisfies (A18). Define $\{v'_n\}_{n=0}^{\infty}$ by $v'_0 = v_0$ and $v'_n = \epsilon_+^{n-1} v_1$ for all $n \geqslant 1$. Note by (A10) and (A4) that

$$\Pi'_0 = (v_0'^{\alpha} - v_1'^{\alpha})p'_0 + 2\mu\delta(1 - \delta)v_1'^{1+\alpha}$$

and

$$\Pi_0 = (v_0^{\alpha} - v_1^{\alpha})p_0 + 2\mu\delta(1 - \delta)v_1^{1+\alpha}.$$

Since $v'_0 = v_0$ and $v'_1 = v_1$, these imply that $\Pi'_0 - \Pi_0 = (v_0^{\alpha} - v_1^{\alpha})(p'_0 - p_0)$. Note that $v_0 - v_1 > 0$. Furthermore, $p'_0 - p_0 = (1 - \delta) \sum_{i=0}^{\infty} \delta^i (v'_{i+1} - v_{i+1})$ by (A1). Observe that $v'_n \geqslant v_n$ for all n, since $\{v_n\}_{n=0}^{\infty}$ satisfies (A18). If $v_{n+1}/v_n < \epsilon_+$ for some $n \geqslant 1$, then $v'_{n+1} > v_{n+1}$, which implies that $\Pi'_0 > \Pi_0$ and contradicts our hypothesis that $\{v_n\}_{n=0}^{\infty}$ solves (A3). *Q.E.D.*

The minimally collusive subgame-perfect equilibrium (with $\Pi_0 > 0$) solves a second optimization problem, analogous to (A3):

$$\min_{\{v_n\}_{n=0}^{\infty}} \Pi_0 \tag{A20}$$

subject to (1), (2), (A3b), and $p_n > 0$ (for all $n \geqslant 0$).

Lemma A5. Suppose (A20) is feasible and $\{v_n\}_{n=0}^{\infty}$ solves (A20). If $\alpha < 1$, then constraint (2) is applicable to period zero, and (2) is satisfied with equality.

Proof. Observe that if (A20) is feasible, there exist feasible $\{v_n\}_{n=0}^{\infty}$ such that $\Pi_0 < 2\mu(1 - \delta)v_0^{1+\alpha}$. Hence, (2) is applicable in period zero. Suppose that $\{v_n\}_{n=0}^{\infty}$ yields slack in the period-zero constraint: define $\{v'_n\}_{n=0}^{\infty}$ by $v'_0 = v_0$ and $v'_n = \beta v_n$ for all $n \geqslant 1$. Then there exists β such that $\{v'_n\}_{n=0}^{\infty}$ is feasible and $\Pi'_0 < \Pi_0$, so $\{v_n\}_{n=0}^{\infty}$ does not solve (A20). *Q.E.D.*

Now consider a variant on optimization problem (A3):

$$\max_{\{v_n\}_{n=0}^{\infty}} \Pi_0, \tag{A21}$$

given p_0, and subject to (1), (2), and (A3b).

Lemma A6. If (A21) has a solution, it is uniquely given by the simple-strategy profile (p_0, ϵ_+).

Proof. We follow literally the same proof we used in Lemmas A1–A4 and Theorem A1. The previous argument hinged on defining alternative sales path $\{v'_n\}_{n=0}^{\infty}$, which improved upon $\{v_n\}_{n=0}^{\infty}$, using (A5) for $k \geqslant 1$. But by (A6), $p'_0 = p_0$, so $\{v'_n\}_{n=0}^{\infty}$ is feasible for the same (A21) as $\{v_n\}_{n=0}^{\infty}$. *Q.E.D.*

Theorem A2. Suppose δ satisfies $\bar{\delta}(\alpha) \leqslant \delta < 1$. If $\alpha \geqslant 1$, then $v_n \equiv v_0 \epsilon_+^n$ is one solution to (A20) (there also exist others), and $\Pi_0 = 2\mu(1 - \delta)v_0^{1+\alpha}$. If $\alpha < 1$, (A20) has a unique solution. It corresponds to the simple-strategy profile (p_0, ϵ_+), where $p_0 = \inf\{p > 0: \Pi(p, \epsilon_+) \geqslant 2p\{v_0^{\alpha} - [p/(1 - \delta)]^{\alpha}\}\}$.

Proof. If $\alpha \geqslant 1$, Lemma A1 implies that $\Pi_n \geqslant 2\mu(1 - \delta)v_n^{1+\alpha}$ for all $n \geqslant 0$. Observe that $v_n \equiv v_0 \epsilon_+^n$ satisfies $\Pi_n = 2\mu(1 - \delta)v_n^{1+\alpha}$ for all $n \geqslant 0$, which necessarily solves (A20).

If $\alpha < 1$, suppose to the contrary that $\{v'_n\}_{n=0}^{\infty}$ solves (A20) and $v'_n \neq \epsilon_+^{n-1} v'_1$ for some $n \geqslant 2$. Define p'_0 and Π'_0 by (A1) and (A2). By Lemma A6 the simple-strategy profile (p'_0, ϵ_+) yields joint profits strictly greater than

Π_0'. Observe that $\partial\Pi_0(p, \epsilon_+)/\partial p > 0$ when $p < \rho(1-\delta)v_0$. Hence, there exists $p_0'' < p_0'$ such that $\Pi_0(p_0'', \epsilon_+) = \Pi_0'$. Furthermore, $2p\{v_0^\alpha - [p/(1-\delta)]^\alpha\}$ is monotonically increasing in p when $p < \rho(1-\delta)v_0$. Hence, (p_0'', ϵ_+) satisfies constraint (2) with slack in period zero. We conclude that there exists $p_0''' < p_0''$ such that (p_0''', ϵ_+) is in the feasible region for (A20) and $\Pi_0(p_0''', \epsilon_+) < \Pi_0'$, a contradiction.

Finally, suppose that $v_n' = \epsilon_+^{n-1}v_1'$ for all $n \geq 2$, but that p_0' does not equal p_0 of the theorem. If $p_0' > p_0$, observe by using Lemma A5 that $p_0' < \rho(1-\delta)v_0$, so that $\Pi_0(p_0, \epsilon_+) < \Pi_0(p_0', \epsilon_+)$. If $p_0' < p_0$, observe by the definition of p_0 that (p_0', ϵ_+) is infeasible. *Q.E.D.*

Appendix B

■ First we show that for quite general distribution functions, an analysis similar to that of the main text is possible. In particular, for any distribution function $F(v)$ satisfying inequality (3) (for some $\alpha > 0$ and $L \geq M > 0$), we develop a function $\epsilon_+(\delta)$ such that $\epsilon_+(\delta)$ implies a subgame-perfect equilibrium whenever the discount factor exceeds δ. The function $\epsilon_+(\delta)$ is defined similarly to the ϵ_+ of Lemma 2, so that it is again the case that maximally collusive joint profits converge rapidly to static monopoly profits.

Suppose that $F(v)$ satisfies (3) and that $F^{-1}(\cdot)$ is well defined. Even if F^{-1} is not defined, one can proceed as below, but the notation becomes more cumbersome. For any $p_0' \geq \underline{v}$ and ϵ $(0 < \epsilon < 1)$, define $\{p_n'\}_{n=1}^{\infty}$ by

$$p_n' - \underline{v} = \epsilon^n(p_0' - \underline{v}). \tag{A22}$$

Let $\{v_n'\}_{n=0}^{\infty}$ denote the sequence of cut-off valuations induced by $\{p_n'\}_{n=0}^{\infty}$. Define $\{v_n\}_{n=0}^{\infty}$ to be the sequence of cut-off valuations that induces the *same sales* on $F(v)$ as $\{v_n'\}_{n=0}^{\infty}$ induced on $L(v-\underline{v})^\alpha$:

$$v_n = F^{-1}[L(v_n' - \underline{v})^\alpha], \qquad \text{for all} \qquad n \geq 0. \tag{A23}$$

Now define $\{p_n\}_{n=0}^{\infty}$ to be a price sequence that induces $\{v_n\}_{n=0}^{\infty}$ by using equation (A1). We shall establish conditions on δ and ϵ such that $\{p_n\}_{n=0}^{\infty}$ is the equilibrium price path of a subgame-perfect equilibrium.

We need to establish that each firm's share of joint profits along the equilibrium starting from period n $(\Pi_n/2)$ exceeds any firm's optimal deviation in period n (denoted Π_n^{dev}), for all $n \geq 1$. (The inequality $\Pi_0/2 \geq \Pi_0^{dev}$ will also be satisfied, unless v_1 is excessively close to $\bar{v}$ or to zero.) Observe that $p_n \geq p_n'$ for all $n \geq 0$ (using (A1) and (3)). Hence,

$$\Pi_n = \sum_{i=0}^{\infty} \delta^i p_{n+i}[F(v_{n+i}) - F(v_{n+i+1})] \geq \sum_{i=0}^{\infty} \delta^i p_{n+i}'[L(v_{n+i}' - \underline{v})^\alpha - L(v_{n+i+1}' - \underline{v})^\alpha]$$

by (A23). Substituting (A22) and using $v_{n+1}' - p_n' = \delta(v_{n+1}' - p_{n+1}')$ eventually yield

$$\Pi_n \geq \frac{L(\epsilon^{-1}-\delta)^\alpha(1-\epsilon^\alpha)(p_n - \underline{v})^\alpha \underline{v}}{(1-\delta)^\alpha(1-\delta\epsilon^\alpha)} + \frac{L(\epsilon^{-1}-\delta)^\alpha(1-\epsilon^\alpha)(p_n-\underline{v})^{1+\alpha}}{(1-\delta)^\alpha(1-\delta\epsilon^{1+\alpha})}. \tag{A24}$$

We also derive an upper bound on Π_n^{dev}. Observe that $\Pi_n^{dev} \leq \max_v \{(1-\delta)v[F(v_n) - F(v)]\}$. The expression in braces may be further bounded[16] by

$$\Pi_n^{dev} \leq (1-\delta)\underline{v}F(v_n) + \max_v \{(1-\delta)(v-\underline{v})[F(v_n) - F(v)]\}. \tag{A25}$$

Using $F(v) \geq M(v-\underline{v})^\alpha$ and performing extensive algebra yield

$$\Pi_n^{dev} \leq \frac{L(\epsilon^{-1}-\delta)^\alpha(p_n-\underline{v})^\alpha \underline{v}}{(1-\delta)^{\alpha-1}} + \left(\frac{L}{M}\right)^{1/\alpha} \frac{L\alpha(\epsilon^{-1}-\delta)^{1+\alpha}(p_n-\underline{v})^{1+\alpha}}{(1+\alpha)^{1+1/\alpha}(1-\delta)^\alpha}. \tag{A26}$$

If the first term of (A24) exceeds twice the first term of (A26) *and* the second term of (A24) exceeds twice the second term of (A26), we have $\Pi_n/2 \geq \Pi_n^{dev}$. Observe that $\mu \leq 1$ and $(L/M)^{1/\alpha} \geq 1$. Hence, a sufficient condition for this is

$$\psi(\epsilon) \equiv \epsilon(1-\epsilon^\alpha) - 2\left(\frac{L}{M}\right)^{1/\alpha}(1-\delta\epsilon)(1-\delta\epsilon^{1+\alpha}) \geq 0. \tag{A27}$$

Observe that, as in Appendix A, there exists $\bar{\delta}(\alpha, L, M) < 1$ such that $\psi(\epsilon)$ has a root in [0, 1] whenever $\delta \geq \bar{\delta}(\alpha, L, M)$. Define $\epsilon_+(\delta) = \sup\{\epsilon \leq 1: \psi(\epsilon) \geq 0\}$. As before, $\epsilon_+(\delta)$ converges to 1 as δ approaches 1, and the convergence is rapid. By choosing v_1 very close to the static monopoly price ρ, we obtain nearly static monopoly profits; by choosing v_1 very close to $\bar{v}$, we obtain nearly zero profits. By continuously varying v_1 from $\bar{v}$ to ρ, we generate all intermediate payoffs.

Now we prove two lemmas that immediately imply Theorem 4.

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[16] The intuition for the bound in (A25) is: the right side of equation (A25) amounts to selecting a discriminatory two-price schedule, where the firm charges $(1-\delta)v$ to all customers with valuation exceeding v and charges $(1-\delta)\underline{v}$ to all other customers. Profits without this price discrimination are necessarily lower.

Lemma A7. For every $0 < s < \infty$ and every $0 < \omega < r/(r+s)$, there exists $\bar{z}_1 > 0$, such that for all $0 < z \leqslant \bar{z}_1$ the path (ω, ϵ) with $\epsilon = e^{-sz}$ is a subgame-perfect equilibrium path after any deviation from path 1.

Proof. After any deviation x in a particular period (which we identify, without loss of generality, as period -1), all customers with valuations $\leqslant v_0$ remain to be served. Assuming that $\omega \leqslant \gamma^{-1} = (1-\delta)/(1-\epsilon\delta) \rightarrow r/(r+s)$, one readily calculates that $v_0 = x/(1-\delta(1-\omega))$ if $x \leqslant (1-\delta(1-\omega))v_{-1}$, and $v_0 = v_{-1}$ otherwise. Since ω represents a *fraction* of the highest remaining consumer valuation v_0, there is, as far as incentive compatibility is concerned, no loss of generality in assuming that $v_0 = 1$. Conditions for (ω, ϵ) to induce a subgame-perfect equilibrium thus coincide with conditions for the simple strategy (ω, ϵ) to give a subgame-perfect equilibrium. The result then immediately follows from the same reasoning we use for Theorem 3. *Q.E.D.*

Lemma A8. For every $0 < s < \infty$, there exist $\omega > 0$ and $\bar{z}_2 > 0$ such that for all $0 < z \leqslant \bar{z}_2$, the monopolist has no incentive to deviate from path 1.

Proof. Confine the choice of ω to $(0, \gamma^{-1})$ so that in the period after the deviation x there are positive sales. If $x \leqslant [1-\delta(1-\omega)]v_{-1}$, profits from deviating will be bounded above by: $\Pi^{dev} \leqslant xv_{-1}^{\alpha} \leqslant [1-\delta(1-\omega)]v_{-1}^{1+\alpha}$. Observe that $\lim_{z\to 0} \Pi^{dev} \leqslant \omega v_{-1}^{1+\alpha}$. If $x \geqslant [1-\delta(1-\omega)]v_{-1}$, profits from deviating will satisfy

$$2\Pi^{dev} = \omega v_{-1}^{1+\alpha}[1-\gamma^{\alpha}\omega^{\alpha}(1-\delta\epsilon)/(1-\delta\epsilon^{1+\alpha})].$$

Again, observe that $\lim_{z\to 0} \Pi^{dev} \leqslant \omega v_{-1}^{1+\alpha}$.

Meanwhile, profits Π_{-1} from continuing on the main path satisfy

$$\Pi_{-1} \geqslant [(1-\delta)\epsilon(1-\epsilon^{\alpha})/(1-\delta\epsilon)(1-\delta\epsilon^{1+\alpha})]\, v_{-1}^{1+\alpha}$$

(with equality unless we are in the initial period). The coefficient of $v_{-1}^{1+\alpha}$ in this inequality approaches $\dfrac{\alpha rs}{(r+s)(r+(1+\alpha)s)}$ as z approaches 0. Hence, for any ω such that $\omega < \min\left\{\dfrac{r}{r+s}, \dfrac{\alpha rs}{(r+s)(r+(1+\alpha)s)}\right\}$, there exists $\bar{z}_2 > 0$ such that the monopolist has no incentive to deviate from path 1 when $0 < z \leqslant \bar{z}_2$. *Q.E.D.*

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