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Efficient Sequential Bargaining

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Suppose that a seller and a buyer have private valuations for a good, and that their respective utilities from a trading mechanism are given by u_s and u_b. (These utilities are determined by the valuation for the good, by whether a trade occurs, and by the price which is paid.) Consider the problem of maximizing $E[\lambda u_s+(1-\lambda)u_b]$ for some weight λ in the unit interval. It is shown in this article that, if λ is sufficiently close to zero or one, then the maximum value of this objective function attainable by a static revelation mechanism can be arbitrarily closely approximated by equilibria of the sequential bargaining games in which only a single player makes offers. That is, the welfare bound implied by the revelation principle is virtually attainable in offer/counteroffer bargaining. The main condition needed for this result is a monotone-hazard-rate assumption about the distribution of types. A class of examples is presented in which the result holds for all λ (i.e. the entire ex ante Pareto frontier).

1. INTRODUCTION

In a remarkable paper, Myerson and Satterthwaite (1983) analyzed static bargaining with two-sided incomplete information, in direct mechanism terms. Two players simultaneously report their valuations to a mediator, who carries out trade with prescribed probabilities and expected payments in a single trading round. Myerson and Satterthwaite characterized ex ante efficient bargaining for fairly general distributions of seller and buyer valuations. Moreover, they demonstrated that, when the supports overlap, ex ante efficiency requires ex post inefficiency: in particular, there exist pairs (s, b) of valuations such that positive gains from trade exist (i.e. $b>s$), yet the players trade with probability zero.

The direct mechanism approach can be criticized on account that, in real life, *traders do not bargain in this way.* First, bargaining better fits a dynamic rather than a static description. For example, after any given round of negotiations, players usually "are unable to commit to walking away from the bargaining table" (Cramton (1984, pp. 579-580)). Second, bargaining typically progresses through successive, rather than simultaneous, moves; actual mediatorless bargaining seldom involves simultaneous bids, and the role of real mediators is quite different from that posited by mechanism design. Thus, much of the bargaining literature has examined extensive form games which are dynamic and infinite-horizon, and in which offers and counteroffers are made successively (Rubinstein (1982)). See Wilson (1987) and Rubinstein (1987) for reviews of this literature.

The belief in the bargaining literature has been that sequentiality comes at the expense of efficiency. Researchers have argued that, if the duration of bargaining is not artificially cut off, then delay is needed to credibly signal valuation. Since trading with delay corresponds (in mechanism terms) to a probability strictly between zero and one, this would imply an efficiency loss relative to the Myerson-Satterthwaite optimum.

The literature also provides good reason to expect that the absence of simultaneous moves would impede information revelation. In direct mechanisms, each player reveals his private information before hearing his opponent's report. By way of contrast, in sequential bargaining, the player who reveals second may be less apt to report truthfully than if he were still ignorant of his opponent's report. To the extent that information revelation is inhibited, this might further contribute to waste.

Thus, the main result of this article may come as a surprise. We analyse the two simplest infinite-horizon, offer/counteroffer extensive forms—the seller-offer and buyer-offer games—for fairly general distributions of valuations. Despite the above considerations as well as the further restriction of only one player making offers, these games admit extremely efficient outcomes. We construct sequential equilibria where essentially all usable information is revealed in the initial two trading rounds, and with the limiting property that trade occurs instantaneously or not at all.[1] In fact, we demonstrate that a portion of the Myerson–Satterthwaite efficient frontier is implementable in the seller-offer game and another portion is implementable in the buyer-offer game; together, the entire Pareto frontier is sometimes obtained.

Consider the well-known example where traders' valuations are both uniformly distributed on the unit interval. Our main theorem, when particularized to this special case, establishes that all ex ante efficient mechanisms which favour the seller (i.e. with

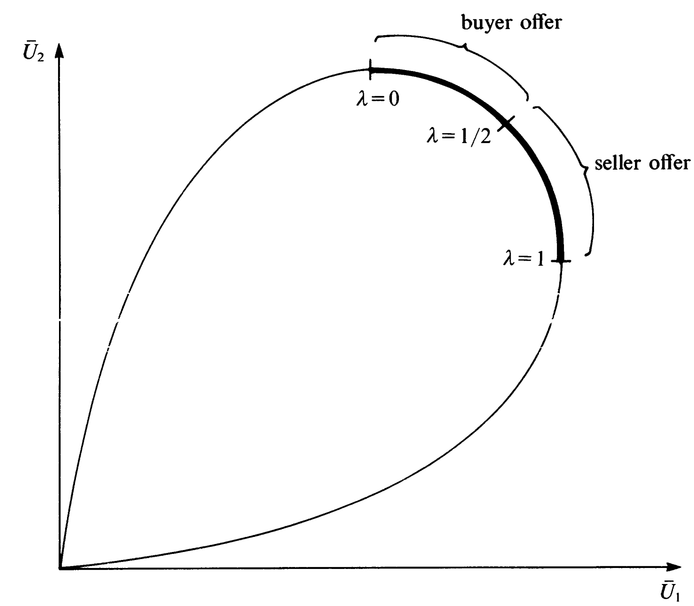


FIGURE 1

Ex ante feasible utility set and portion of the Pareto frontier implementable in the seller- and buyer-offer game (where $\bar{U}_i = \int_0^1 U_i(z) f_i(z)dz$).

1. The limiting properties that information is revealed instantaneously and trade occurs immediately or not at all is somewhat reminiscent of results by Gul, Sonnenschein and Wilson (1986) and Gul and Sonnenschein (1988) for *one-sided* incomplete information. In those earlier papers, information is revealed instantaneously and trade inevitably occurs immediately, yielding efficiency for every welfare weight favouring the informed party.

However, the limiting properties come about in those earlier papers for very different reasons. There, the limiting properties are derived as implications of stationarity assumptions. (Removing the stationarity assumptions eliminates the predisposition for instantaneous revelation and trade. At the same time, it permits efficiency for weights favouring the uninformed party (Ausubel and Deneckere (1989*a*, *b*)).

seller weight $\lambda \in [1/2, 1]$) are implementable in the seller-offer game. Symmetrically, all efficient mechanisms favouring the buyer (i.e. $\lambda \in [0, 1/2]$) are implementable in the buyer-offer game. (See Figure 1.) Thus, the trading rule that maximizes ex ante expected gains from trade (i.e. $\lambda = 1/2$), which is known to be an equilibrium of the sealed-bid double auction (Chatterjee and Samuelson (1983)), does not require the availability of simultaneous moves. Even more surprisingly, its implementation does not even depend on the ability of both players to make bids.

Successive exchanges of offers are probably the oldest and still most commonly used bargaining institutions. As economists, we should ask whether "these institutions survive because they employ trading rules that are efficient for a wide class of environments" (Wilson (1987, p. 37)). Our main theorem suggests that this may indeed be the case.[2] For fairly general distributions, offer/counteroffer games may collectively be viewed as unbiased institutions: changing the relative weights attached to traders, or changing the distributions of their valuations, does not require an alteration of the rules of the games. Rather, traders can simply play different equilibria of the same trading institutions and continue to realize efficiency.[3] In fact, it is sometimes the case that the two simplest offer/counteroffer extensive forms (where a single party has the exclusive ability to make offers) are sufficient to "span" the Pareto frontier.

It is illuminating to contrast the robustness of the seller- and buyer-offer games with the non-robustness of the sealed-bid double auction. When the buyer's bid p_b exceeds the seller's bid p_s, let trade occur at $kp_b + (1-k)p_s$. Fixing $k = 1/2$, this double auction implements efficiency under equal weighting and uniform distributions. However, efficiency for $\lambda \neq 1/2$ requires a change in k, and efficiency for other distribution functions generally requires other modifications in the trading institution.

Two other points should be emphasized. First, while efficient sequential bargaining is possible, tremendously inefficient sequential equilibria also exist.[4] Second, despite the multiplicity of equilibria, this is *not* a realm for folk theorems. There exist many static mechanisms which are not implementable in various offer/counteroffer games.[5]

The article is structured as follows. Section 2 defines the sequential game and static mechanism concepts. Section 3 describes the main result. Section 4 formulates the scheme whereby we partially pool different seller types and explains why we use a nonconvex information revelation structure. Section 5 develops the limiting concept of the two-price mechanism and explains its relationship with sequential equilibrium. Section 6 concludes. Various proofs are relegated to two Appendices.

2. We would like to acknowledge our debt to Robert Wilson for suggesting this interpretation of our result.

3. Some caveats should be added to the conclusions of this paragraph. First, offer/counteroffer extensive forms also permit very inefficient equilibria. We lack any theory to explain why players would select efficient equilibria over inefficient equilibria. Second, our main theorem establishes that ex ante efficient mechanisms can be approximated only for welfare weights in neighbourhoods of zero or one, whereas it seems more plausible that players would gravitate toward optima for welfare weights closer to one-half. However, the examples in Section 5 suggest that efficient mechanisms can also be approximated for middle-range λ's.

4. In a previous paper (Ausubel and Deneckere (1988*a*)), we demonstrated the existence of a large class of equilibria. Moreover, our No Trade Theorem (Ausubel and Deneckere (1992, Theorem 1)) proves that equilibria with the Coase Conjecture property converge to maximal inefficiency as the interval between offers approaches zero.

5. In Lemma 4.3 below, we demonstrate that $p(\cdot, \cdot)$ is *not* implementable in any offer/counteroffer game if $p(\cdot, \cdot)$ is not "balanced". Furthermore, $p^4(\cdot, \cdot)$ in Ausubel and Deneckere (1988*a*) provides an example of a balanced mechanism which is *not* implementable in the seller-offer game. Counterexamples to general feasibility theorems will be the subject of a future paper.

The absence of folk theorems in two-sided incomplete information contrasts with our previous (1989*b*) results on one-sided incomplete information. The seller-offer game there implements the entire set of ex post individually rational ICBM's.

2. THE MODEL

Two parties, a seller and a buyer, are negotiating over the price of a single indivisible object worth s to the seller and b to the buyer. At the time the bargaining commences, each trader is aware of his own valuation, but treats his opponent's as a random variable. These random variables are distributed independently on the common interval $[0, 1]$, according to the (commonly-known) distribution functions $F_1(s)$ and $F_2(b)$. Each distribution function $F_i(\cdot)$ possesses a density $f_i(\cdot)$ which is assumed to be continuous and strictly positive everywhere on the support. Traders are interested in maximizing their expected monetary gain.

A. *Incentive compatible bargaining mechanisms*

A *bargaining mechanism* is a game in which both parties simultaneously report their reservation prices to a mediator, who then determines whether the good is transferred, and how much the buyer is to pay the seller. A bargaining mechanism is completely characterized by the two outcome functions, $p(\cdot, \cdot)$ *and* $x(\cdot, \cdot)$, where $p(s, b)$ denotes the probability of trade given the reports of s and b, and $x(s, b)$ denotes the expected payment. With every bargaining mechanism $\{p, x\}$, we associate:

$$\begin{aligned} \bar{p}_1(s) &= \int_0^1 p(s, v_2) f_2(v_2)\, dv_2 & \bar{p}_2(b) &= \int_0^1 p(v_1, b) f_1(v_1)\, dv_1 \\ \bar{x}_1(s) &= \int_0^1 x(s, v_2) f_2(v_2)\, dv_2 & \bar{x}_2(b) &= \int_0^1 x(v_1, b) f_1(v_1)\, dv_1, \end{aligned} \tag{2.1}$$

where $\bar{p}_1(s)$ is the probability of agreement and $\bar{x}_1(s)$ is the expected revenue to the seller of type s, and $\bar{p}_2(b)$ is the probability of agreement and $\bar{x}_2(b)$ is the expected payment for the buyer of type b. Thus, the seller's and buyer's (interim) expected payoffs are given by:

$$U_1(s) = \bar{x}_1(s) - s\bar{p}_1(s) \quad \text{and} \quad U_2(b) = b\bar{p}_2(b) - \bar{x}_2(b). \tag{2.2}$$

A bargaining mechanism is *incentive compatible* if all player types have an incentive to report truthfully, i.e.

$$U_1(s) \geqq \bar{x}_1(s') - s\bar{p}_1(s'), \qquad U_2(b) \geqq b\bar{p}_2(b') - \bar{x}_2(b'), \tag{2.3}$$

for all s, s', b and b' in $[0, 1]$. A mechanism $\{p, x\}$ is *individually rational* if all player types want to participate voluntarily, i.e. $U_1(s) \geqq 0$ and $U_2(b) \geqq 0$, for every $s, b \in [0, 1]$. Mechanisms satisfying both the individual rationality and incentive constraints will be referred to as *incentive compatible bargaining mechanisms* (ICBM's).

B. *Offer/counteroffer bargaining games*

Mechanisms are valuable devices for analysing the incentive constraints present in bargaining. However, the usual description of bargaining, rather than being static and direct in nature, involves players making repeated offers and counteroffers through time until an agreement is concluded, or an impasse is reached. By an *offer/counteroffer game*, we will mean any extensive form game (finite or infinite) in which at discrete moments in time one (and only one[6]) of the players is given an opportunity to make an offer, which

6. We specifically rule out extensive forms which permit simultaneous moves since, even in the one-shot complete information case, these permit any division of the surplus.

the other player can then either accept or reject. Acceptances always conclude the bargaining; rejections may or may not do so. We also assume that the time between successive bargaining rounds is bounded away from zero, and that players discount the future with a common interest rate, r. Hence, if the good is traded at time t for the price ϕ, the seller obtains a surplus of $e^{-rt}(\phi - s)$, and the buyer a surplus of $e^{-rt}(b-\phi)$. Note that implicit in the definition of offer/counteroffer game is the assumption that money only changes hands if the good changes hands. In particular, the rules of the game do not permit players to pay each other history-contingent payments unrelated to the accepted offer. One example of an offer/counteroffer game is the *seller-offer game*, in which the seller gets to make all the offers at discrete moments in time, spaced z apart, and the bargaining continues indefinitely until an offer is accepted. A symmetric example is the *buyer-offer game*, in which the buyer gets to make all the offers.

The revelation principle implies that to every Nash equilibrium of an offer/counteroffer extensive form there corresponds a *sequential bargainig mechanism.*[7] Such a mechanism specifies a pair of outcome functions $t(\cdot,\cdot)$ and $x(\cdot,\cdot)$, where $t(s,b)$ denotes the time at which the good will be traded and $x(s,b)$ the discounted expected payment from the buyer to the seller, given respective reports of s and b. Letting $p(s,b)=e^{-rt(s,b)}$, we see that every Nash equilibrium maps into an ICBM $\{p,x\}$. Similarly, every sequential equilibrium of an offer/counteroffer game induces an ICBM after any history of the game. Viewed from this perspective, static mechanisms thus serve as useful vehicles for studying the outcomes of dynamic bargaining models.

C. Implementation of direct mechanisms in sequential games

In this article, we will investigate whether ex ante efficient static mechanisms can be arbitrarily closely approximated by sequential equilibria of offer/counteroffer games (see also Ausubel and Deneckere (1989*b*, 1992)). Formally, our notion of implementation may be stated as:

Definition 2.1. Let $p(\cdot,\cdot)$ be associated with an ICBM. We will say that p is *implementable by sequential equilibria of the seller-offer game* if there exists a sequence $\{\sigma^n, z^n\}_{n=1}^{\infty}$ such that:

(i) $z^n \downarrow 0$ and, for every $n \geqq 1$, σ^n is a sequential equilibrium of the seller-offer game where the time between offers is z^n; and
(ii) If $m(\cdot)$ denotes the $F_1 \times F_2$-measure on $[0,1]\times[0,1]$, and if $p^n(\cdot,\cdot)$ denotes the probability of trade function induced by σ^n, then for all $\varepsilon>0$:

$$m\{(s,b): |p^n(s,b)-p(s,b)|>\varepsilon\}\to 0,$$

as $n\to\infty$.

Implementation in the buyer-offer game is defined analogously. When every σ^n of the implementing sequence is stationary in the sense that history matters only insofar as it is reflected in current beliefs, then we will say that p is *implementable by stationary sequential equilibria.*

Our definition embodies a relatively weak notion of ex post implementation, as it requires only convergence *in probability* of the probability-of-trade function $p(\cdot,\cdot)$.

7. The terminology here is borrowed from Cramton (1985). Note that our language below assumes the typical case in which randomization of outcomes over time does not occur.

Moreover, the payment function $x(\cdot,\cdot)$ is not required to be matched at all. Nevertheless, our definition implies a very strong notion of interim implementation: the traders' interim utilities, $U_1(\cdot)$ and $U_2(\cdot)$, are required to converge uniformly. (For a proof, see Ausubel and Deneckere (1988*a*, Proposition 1).)

3. THE MAIN RESULT

Perhaps the most important question concerning dynamic bargaining games is whether sequential rationality necessarily implies greater inefficiency than is mandated by the requirements of (static) incentive compatibility and individual rationality. A static mechanism $\{p, x\}$ is defined to be *ex ante efficient* if it maximizes $\lambda \int_0^1 U_1(s)dF_1(s)+(1-\lambda)\int_0^1 U_2(b)dF_2(b)$, over all ICBM's, for some weight $\lambda \in [0, 1]$. The main result of this article is that a portion of the ex ante Pareto frontier of mechanisms (and, for some examples, the entire Pareto frontier) can be (approximately) replicated by sequential equilibria of offer/counteroffer games. In fact, this conclusion remains true even if attention is restricted to bargaining games in which only one of the parties is permitted to make offers. We have:

Theorem 3.1. *If $F_1(s)/f_1(s)$ and $[F_2(b)-1]/f_2(b)$ are strictly increasing functions, then*:

(i) *there exists $\lambda_s \in (0, 1)$ such that, for every $\lambda \in [\lambda_s, 1]$, the ex ante efficient mechanism which places weight λ on the seller is implementable in the seller-offer game*; *and*
(ii) *there exists $\lambda_b \in (0, 1)$ such that, for every $\lambda \in [0, \lambda_b]$, the ex ante efficient mechanism which places weight λ on the seller is implementable in the buyer-offer game.*

According to our definition of "implementability", Theorem 3.1 would require that there exists a sequence of equilibria whose outcomes (expressed as probabilities of trade between seller and buyer types) converge in measure to efficient mechanisms. Now it is known (Myerson and Satterthwaite (1983) and Williams (1987) that, under our distributional assumptions, efficient mechanisms satisfy what could be called the *Northwestern Criterion.* First, they are *0–1 mechanisms*: each pair of seller-buyer types trades either with a probability of zero or a probability of one. Second, there exists a *boundary* given by the equation $b = g(s)$, which is continuous and strictly increasing on the interval $[0, \hat{s}]$, with the property that a seller-buyer pair trades with probability one if and only if $\{s, b\}$ is northwest (i.e. above and to the left) of the boundary. (For $s \in (\hat{s}, 1]$, $g(s) = 1$.) Thus, an equilibrium along a sequence which implements an efficient mechanism should have the property that most seller-buyer pairs to the northwest of the mechanisms's boundary trade very near the start of the game, whereas those to the southeast of the boundary trade indefinitely far into the future or not at all.

For example, consider the well-known Chatterjee-Samuelson (1983) mechanism:

$$p(s, b) = \begin{cases} 1 \\ 0, \end{cases} \qquad x(s, b) = \begin{cases} (s+b+1/2)/3 & \text{if } b \geqq s+1/4, \\ 0 & \text{otherwise,} \end{cases} \tag{3.1}$$

where s and b are uniformly distributed on $[0, 1]$. Myerson and Satterthwaite (1983) established that this mechanism maximizes expected total gains from trade over all ICBM's (i.e. it is ex ante efficient under equal weighting of seller and buyer ($\lambda = 1/2$)). Observe that $\hat{s}$ in this example is $3/4$, and the boundary $g(\cdot)$ is given by $g(s) = s + 1/4$.

One could naively attempt to implement an efficient mechanism in the seller-offer game by proposing an equilibrium in which, roughly speaking, the seller of type s fully separates in the initial period by charging a price of $g(s)$ (and then continues to inflexibly charge $g(s)$ in all subsequent periods). This construction yields the required probability-of-trade function $p(\cdot, \cdot)$, as every buyer type $b \geqq g(s)$ would transact with seller s in the initial period, while every buyer type $b < g(s)$ would never transact. (Indeed, such a construction would be analogous to our approach in Ausubel and Deneckere (1988*a*), where we demonstrated that "sequentially seller-first mechanisms" can be implemented in the seller-offer game.)

However, the naive construction is fatally flawed, in that (for efficient mechanisms) it yields an expected-payment function which is different from $\bar{x}_1(\cdot)$ (as defined in (2.1)).[8] The reader should observe that the expected payment received by seller s in the proposed equilibrium would equal $g(s)\bar{p}_1(s)$, since $\bar{p}_1(s)$ represents the probability that the buyer's type exceeds $g(s)$ and the payment equals $g(s)$ in this event (zero, otherwise). For the Chatterjee-Samuelson example, this expected payment equals $(1/4+s)(3/4-s)$. In contrast, $\bar{x}_1(s)$ can also be calculated directly from the mechanism which we are attempting to implement, by integrating $x(s, b)$ over all b such that $g(s) \leqq b \leqq 1$. For the Chatterjee-Samuelson example, integrating (3.1) yields $\bar{x}_1(s) = (3/8 + s/2)(3/4 - s)$. These two formulae do *not* match for $s \in [0, 3/4]$, except at the single point $s^* \equiv 1/4$. A more general argument in Section 4 excludes a naive implementation of all efficient mechanisms (where any welfare weight $\lambda \neq 1$ is placed on the seller).[9]

Intuitively speaking, what fails in the naive construction is that seller types $s \in (0, s^*)$ receive *too little* payment relative to what is required for incentive compatibility, whereas $s \in (s^*, \hat{s})$ receive *too much* payment.[10] In order to remedy this defect, we need to find a means to enable seller types in $(s^*, \hat{s})$ to effectively *cross-subsidize* those in $(0, s^*)$. The need for cross-subsidization motivates the following *pairwise-pooling scheme.*

Let s^* denote the seller type strictly between 0 and $\hat{s}$ for whom $g(s)\bar{p}_1(s) = \bar{x}_1(s)$: the existence and uniqueness of s^* will be demonstrated via Lemma 4.2 and equation (4.6); as noted above, $s^* = 1/4$ for the Chatterjee-Samuelson mechanism. Let $s^* \equiv c_0 < c_1 < \cdots < c_{N-1} < c_N \equiv \hat{s}$ be a grid of seller types partitioning the interval $[s^*, \hat{s}]$ into N sub-intervals, and let $0 \equiv a_0 < a_1 < \cdots < a_{N-1} < a_N \equiv s^*$ be a corresponding grid of seller types partitioning the interval $[0, s^*]$ into N sub-intervals.

The equilibrium strategies are as follows. In period zero, seller types belonging to the paired intervals $[a_k, a_{k+1})$ and $[c_k, c_{k+1})$ *pool* by charging the same initial price $\hat{p}_0(a_k)$. This initial offer reveals to the buyer that $s \in [a_k, a_{k+1}) \cup [c_k, c_{k+1})$, but not whether $s \in [a_k, a_{k+1})$ or $s \in [c_k, c_{k+1})$. In period one, seller types further separate: a seller with $s \in [a_k, a_{k+1})$ reduces her price to $\hat{g}(a_k)$, while a seller with $s \in [c_k, c_{k+1})$ raises her price to $\hat{g}(c_k)$. With the exception of seller types in the lowest pool,[11] the seller continues to charge the same price, $\hat{g}(a_k)$ or $\hat{g}(c_k)$, in all subsequent periods. Finally, seller types belonging to $[\hat{s}, 1]$ always make non-serious offers of one.

The buyer forms expectations and optimizes, subject to this seller behaviour. Define:

$$\theta_k \equiv \frac{F_1(a_{k+1}) - F_1(a_k)}{[F_1(a_{k+1}) - F_1(a_k)] + [F_1(c_{k+1}) - F_1(c_k)]}. \tag{3.2}$$

8. Since $\bar{x}_1(s)$ is the unique interim expected payment which, when paired with $\bar{p}_1(s)$, is incentive compatible, one must conclude that the naive construction violates incentive compatibility for the seller.

9. When $\lambda = 1$, the efficient mechanism is the "monopoly mechanism". In Ausubel-Deneckere (1992) we show that this mechanism is implementable in the seller-offer game by the "naive construction" of the main text. The reason that we are able to demonstrate this is that, when $\lambda = 1$, $g(s)\bar{p}_1(s)$ coincides with $\bar{x}_1(s)$, for all $s \in [0, 1]$.

10. More formally, $g(s)\bar{p}_1(s) < \bar{x}_1(s)$ for seller types $s \in (0, s^*)$, whereas $g(s)\bar{p}_1(s) > \bar{x}_1(s)$ for $s \in (s^*, \hat{s})$.

11. A seller whose valuation belongs to $[a_0, a_1)$ instead follows a descending price path: charging a price of $e^{-\lambda(m-1)z}g(a_0)$ in all periods $m \geqq 1$ that this price exceeds her valuation, and making non-serious offers of one thereafter. This qualitatively-different behaviour is necessitated by the fact that, in any sequential equilibrium of the seller-offer game, the price charged by the zero-valuation seller must converge to zero. A more detailed description of the treatment of the bottom pool is provided in Section 4 of Ausubel and Deneckere (1992).

Then, after the seller's period-zero offer, the buyer should assign probability θ_k to $s \in [a_k, a_{k+1})$ and probability $(1-\theta_k)$ to $s \in [c_k, c_{k+1})$. Furthermore, one can determine a threshold valuation, $\hat{\beta}(a_k)$, such that any buyer with valuation exceeding $\hat{\beta}(a_k)$ accepts the seller's initial (pooling) offer and any lower buyer type rejects. After the seller's period-one offer, the buyer should believe for sure that $s \in [a_k, a_{k+1})$ (or $s \in [c_k, c_{k+1})$). Any buyer with valuation b exceeding $\hat{g}(a_k)$ (or $\hat{g}(c_k)$) accepts the seller's next offer—since price will never drop again—and any buyer with lower valuation rejects that and all subsequent offers.[12]

The seller is deterred from reducing her price below that specified in the equilibrium (in an effort to generate additional sales) by the prospect of adverse inferences and hence expectations of still lower prices. More precisely, let σ be a weak-Markov equilibrium in the corresponding seller-offer game *of one-sided incomplete information* in which the seller's valuation is commonly known to equal zero (but the buyer's valuation continues to be distributed according to $F_2(\cdot)$). If the seller ever deviates *detectably* from her equilibrium strategy, the buyer responds by updating his beliefs to $s=0$ and maintains these beliefs forever after. Furthermore, the buyer then expects the weak-Markov equilibrium σ to be played in the continuation.[13] Thus, following a detectable seller deviation, the buyer accepts or rejects using his strategy from σ; and the seller optimizes against the buyer's strategy. Since weak-Markov equilibria display the "Coase Conjecture" property that the initial price converges to zero as the time interval between periods becomes arbitrarily short,[14] this effectively deters the seller from ever deviating detectably.[15] Finally, if the seller ever deviates *undetectably*, she then follows whatever price path maximizes her payoff subject to keeping the deviation undetectable to the buyer. (Whenever that would involve pricing below cost, the seller starts making non-serious offers instead.)[16]

It should further be observed that the above equilibria are not only sequential but also stationary. If two different histories lead to the same pair of beliefs, then the ensuing equilibrium actions are also the same. Similarly, updating rules depend on the history only insofar as it is reflected in the current pair of beliefs.

The pairwise-pooling equilibrium construction is depicted in Figure 2, as applied to the Chatterjee–Samuelson mechanism of equation (3.1). Each of the intervals $[0, s^*)$ and $[s^*, \hat{s})$ is divided into four sub-intervals, so that $N=4$. In panel (a) are illustrated $\hat{p}_0(\cdot)$ and $\hat{g}(\cdot)$, the prices charged in periods zero and one, respectively. Observe that $\hat{p}_0$ is constant-valued on the set $[a_k, a_{k+1}) \cup [c_k, c_{k+1})$, for $k=0, 1, 2, 3$. Moreover, $\hat{g}(\cdot)$ is a step-function approximation of $g(s)=s+1/4$. In panel (b) is illustrated $\hat{\beta}(\cdot)$, the lowest valuation buyer who accepts the period-zero offer, together with $\hat{g}(\cdot)$. The probability

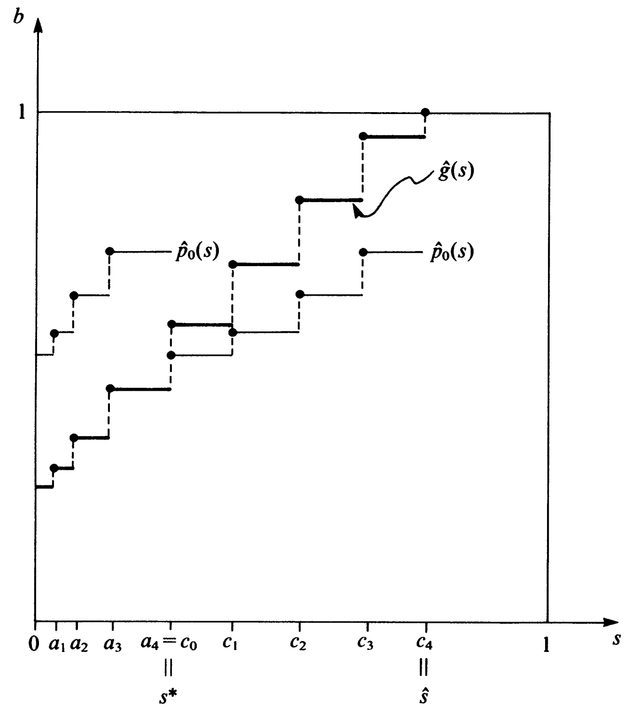
12. For $s \in [a_0, a_1)$, on account of the qualitatively-different seller strategy indicated in the previous footnote, the buyer selects a period in which to purchase taking account of both time impatience and the probability that the seller will cease to make serious offers in subsequent periods.

13. We introduced the approach of utilizing reversion to weak-Markov equilibria in our earlier study of reputation in bargaining and durable goods monopoly (Ausubel and Deneckere (1989*a*)).

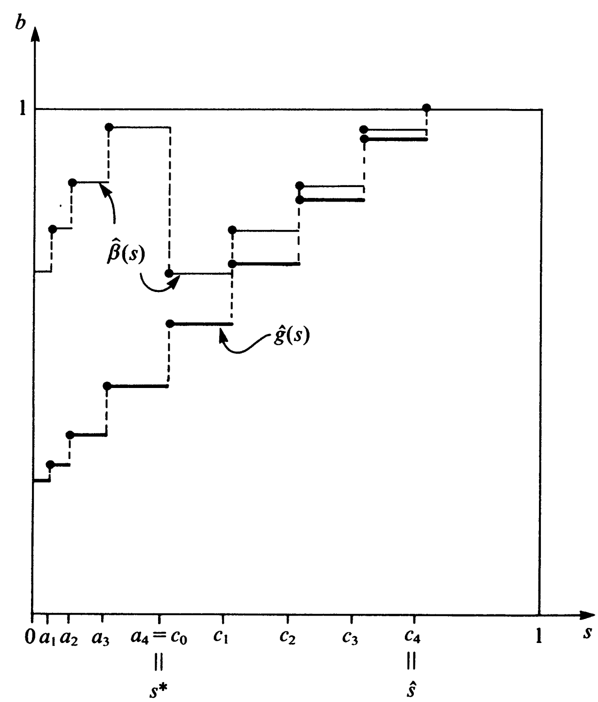
14. The existence of a weak-Markov equilibrium is guaranteed by Proposition 2 of Fudenberg, Levine and Tirole (1985). That it satisfies a uniform version of the Coase Conjecture is shown by Theorem 5.4 of Ausubel and Deneckere (1989*a*).

15. This does not immediately conclude the argument, as the seller's continuation profit in the equilibrium may also be infinitesimally small or even zero. However, following the proof of Facts 2 and 3 in Section 4 of the companion article (Ausubel and Deneckere (1992)), it can be shown that when the time interval is sufficiently short, this does in fact deter the seller from detectably deviating off the equilibrium path.

16. It should be observed that, in this specification, there is no such thing as a history in which there has been a (detectable) buyer deviation. Since buyer valuations extend all the way down to zero, *every* equilibrium offer has a positive probability of rejection. Hence, a buyer rejection can never be an observable deviation. Meanwhile, we need not concern ourselves with what follows a buyer acceptance, as it immediately ends the game.

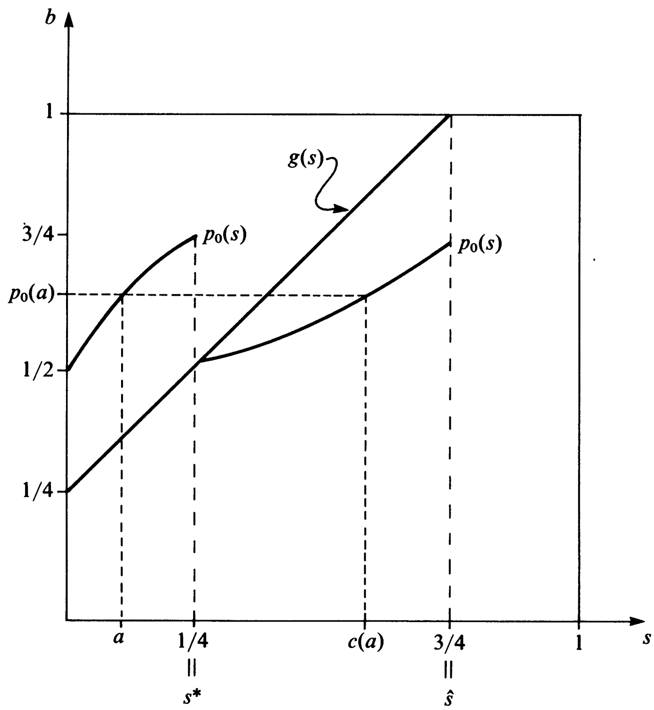


Panel (a)
First and second period prices

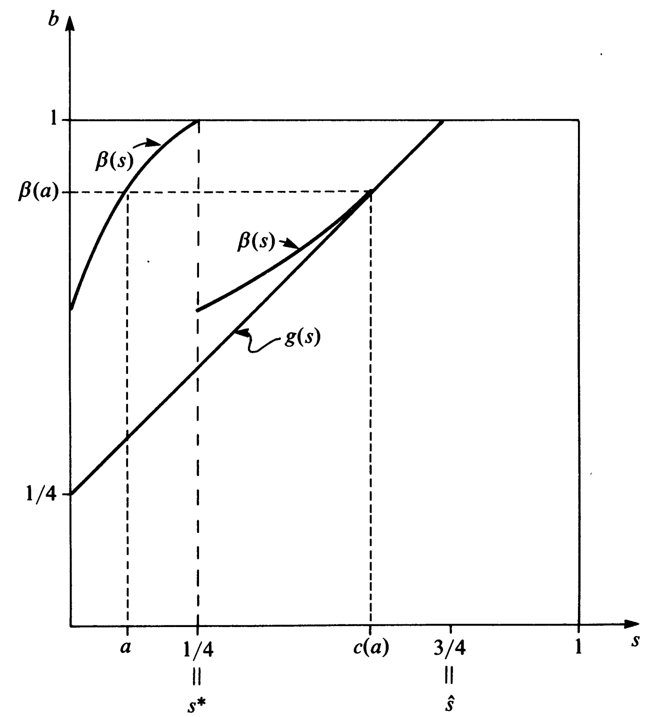


Panel (b)
First and second period sales

FIGURE 2
An illustration of our equilibrium for the Chatterjee-Samuelson mechanism ($N = 4$).



Panel (a)
First and second period prices



Panel (b)
First and second period sales

FIGURE 3

An illustration of the limiting version of our equilibrium for the Chatterjee–Samuelson mechanism.

of trade for each seller-buyer pair can thus be read directly from panel (b): if $s \in [a_k, a_{k+1})$, then $p(s, b) = 1$ for $\hat{\beta}(a_k) \leqq b \leqq 1$, as b accepts the initial offer of type s; $p(s, b) = \delta \equiv e^{-rz}$ for $\hat{g}(a_k) \leqq b < \hat{\beta}(a_k)$, as b rejects the initial offer but accepts the following offer of type s; and $p(s, b) = 0$ for $0 \leqq b < \hat{g}(a_k)$, as b rejects all offers of type s.[17] (An analogous statement holds if $s \in [c_k, c_{k+1})$.) Also observe that $\hat{\beta}(a_k) = \hat{\beta}(c_k)$, for $k = 0, 1, 2, 3$, as the buyer cannot yet distinguish in period zero whether $s \in [a_k, a_{k+1})$ or $s \in [c_k, c_{k+1})$.

As the grid of sub-intervals is made arbitrarily fine and as discounting becomes negligible (i.e. as $N \to \infty$ and $\delta \to 1$), the step functions $\hat{p}_0(\cdot)$, $\hat{g}(\cdot)$ and $\hat{\beta}(\cdot)$ of Figure 2 can be chosen so as to converge uniformly to the piecewise-continuous functions $p_0(\cdot)$, $g(\cdot)$ and $\beta(\cdot)$ depicted in Figure 3. The limiting pooling price, $p_0(\cdot)$, takes equal values on pairs of points: for each $a \in [0, s^*)$, there is an associated $c \in [s^*, \hat{s})$ such that a and c pool in the initial period. The function $g(\cdot)$ is the boundary of the 0–1 mechanism which we are trying to implement (in this example, $g(s) = s + 1/4$). The two functions are linked by the fact that $p_0(s)$ equals the expected value of $g(s)$, conditional on the event that s equals either a or c. Finally, in the limit as $\delta \to 1$, the probability of trade equals one regardless of whether $b \in [\beta(s), 1]$ or whether $b \in [g(s), \beta(s))$, so the probability of trade which can be read from Figure 3(b) corresponds to the efficient mechanism. We thus conclude that the indicated equilibria implement the efficient mechanism in the seller-offer game: the probability-of-trade function from Figure 2(b) converges in measure to the efficient $p(\cdot, \cdot)$ of Figure 3(b) and equation (3.1).[18]

At this point, one crucial issue has not yet been addressed. While we have argued that the naive, fully-separating construction is not incentive compatible, we have not given any indication why the more complex, pairwise-pooling scheme should lead to correct self-selection in the initial period. First, we need to demonstrate how the paired seller types, a_k and c_k, are selected so that a cross-subsidy is paid by each seller type in $(s^*, \hat{s})$ to an appropriate counterpart in $(0, s^*)$. Second, we need to establish how the functions $\hat{p}_0(\cdot)$, $\hat{g}(\cdot)$ and $\hat{\beta}(\cdot)$ are determined so that the cross-subsidies are paid in the correct amount to induce correct self-selection. Third, we need to assure (for internal consistency) that $\hat{\beta}(\cdot) \geqq \hat{g}(\cdot)$. These complicated matters will be treated in the next two sections.

4. THE PAIRING OF TYPES

In this section, we formulate a scheme for pairing together seller types. We define a function $B_1(\cdot)$ which can be calculated for any incentive compatible bargaining mechanism. The function $B_1(\cdot)$ will become a recipe for pairing seller types; we pool $a \in [0, s^*)$ with $c \in [s^*, \hat{s})$ according to the formula:

$$B_1(a) + B_1(c) = B_1(s^*), \tag{4.1}$$

The definition of $B_1(\cdot)$ will guarantee that $B_1(\cdot)$ is continuous and $B_1(0) = 0$. In order for equation (4.1) to unambiguously determine a one-to-one and onto mapping between the intervals $[0, s^*)$ and $[s^*, \hat{s})$, we must further assure that: first, $B_1(\cdot)$ is strictly

17. Obviously, the probabilities are somewhat different from what is described here when $s \in [a_0, a_1)$ and $b < \beta(a_0)$, due to the descending price path followed by seller types in the lowest sub-interval. This different description is consciously ignored both in the text here and in Figure 2, in order to simplify the exposition.

18. As in several of the previous footnotes, we should reiterate that for $s \in [a_0, a_1)$, the probability-of-trade function takes somewhat different values than are indicated in Figure 2(b), as the path of prices offered is *declining*. However, as the time interval z between offers approaches zero, it becomes possible to make the descending price path *arbitrarily flat*, and so the discrepancy with Figure 2(b) converges to zero (satisfying the definition of implementability).

quasiconcave on the domain $[0, \hat{s}]$, attaining a maximum at $s=s^*$; and, second, that $B_1(\hat{s})=0$. These two additional conditions do *not* generally hold for all ICBM's. However, in Lemma 4.2 of this section, we establish that they are satisfied if the mechanism is *ex ante efficient.* It will thus be possible to use the $B_1(\cdot)$ function to define the paired values a_k and c_k used in proving the main theorem.

We define the function $B_1(\cdot)$ and its analogue for buyer types, $B_2(\cdot)$, by:

$$B_1(s)=F_1(s)U_1(s)-G_1(s), \quad \text{and} \quad B_2(b)=[1-F_2(b)]U_2(b)-G_2(b), \tag{4.2}$$

where:

$$G_1(s)=\int_0^s\int_0^1 \{[v_2-(1-F_2(v_2))/f_2(v_2)] \\ -[v_1+F_1(v_1)/f_1(v_1)]\}p(v_1, v_2)f_2(v_2)f_1(v_1)dv_2dv_1$$

$$G_2(b)=\int_b^1\int_0^1 \{[v_2-(1-F_2(v_2))/f_2(v_2)] \\ -[v_1+F_1(v_1)/f_1(v_1)]\}p(v_1, v_2)f_1(v_1)f_2(v_2)dv_1dv_2.$$

The functions $B_1(\cdot)$ and $B_2(\cdot)$ have straightforward interpretations for 0–1 mechanisms with continuous boundaries, as the following lemma indicates:[19]

Lemma 4.1. *For any* 0–1 *mechanism* $\{p, x\}$ *with continuous boundary* $g(\cdot)$, $B_1(s)$ *represents the (weighted) aggregate cross-subsidy required to be received by all seller types in the interval* $[0, s]$, *i.e.*

$$B_1(s)=\int_0^s [\bar{x}_1(v_1)-g(v_1)\bar{p}_1(v_1)]f_1(v_1)dv_1. \tag{4.3}$$

Analogously, $B_2(b)$ *represents the (weighted) aggregate cross-subsidy required to be paid by all buyer types in the interval* $[b, 1]$, *i.e.*

$$B_2(b)=\int_b^1 [\bar{x}_2(v_2)-g^{-1}(v_2)\bar{p}_2(v_2)]f_2(v_2)dv_2. \tag{4.4}$$

Proof. For any 0–1 mechanism with continuous boundary $g(\cdot)$, observe from the definition in (4.2) that $B_1(\cdot)$ is a continuously-differentiable function. Using the fact that $U_1'(s)=-\bar{p}_1(s)$ (see Myerson and Satterthwaite (1983, Theorem 1)), we may compute that:

$$B_1'(s)/f_1(s)=U_1(s)-\int_0^1 [v_2-s-(1-F_2(v_2))/f_2(v_2)]p(s, v_2)f_2(v_2)dv_2. \tag{4.5}$$

Define $\pi(\phi, s)=[\phi-s][1-F_2(\phi)]$, the expected profit to a seller of type s from charging a take-it-or-leave-it price of ϕ. Observing that $(\partial\pi/\partial\phi)(v_2, s)=1-F_2(v_2)-(v_2-s)f_2(v_2)$, and recalling that p is a 0–1 mechanism with boundary $g(\cdot)$, equation (4.5) reduces to:

$$B_1'(s)/f_1(s)=U_1(s)-[g(s)-s][1-F_2(g(s))]=\bar{x}_1(s)-g(s)\bar{p}_1(s), \tag{4.6}$$

for all $s\in[0, 1]$, implying (4.3). Equation (4.4) is derived analogously. ‖

19. Lemma 4.1 has a generalization for non-0-1 mechanisms. If $p(s,\cdot)$ is monotone in b for each s, then $p(s,\cdot)$ uniquely defines a "naive" price path for seller s. Along this path, the net present value of buyer b's payment to s is given by $w(s, b)\equiv\int_0^b r d\mu_s(r)$, where $\mu_s(r)\equiv p(s, r)$. (See Ausubel and Deneckere (1989*b*, Theorem 1).) It can then be shown that $B_1'(s)/f_1(s)$ still represents the cross-subsidy to seller s relative to this naive income stream, i.e. $B_1'(s)/f_1(s)=U_1(s)-\int_0^1 w(s, v_2)f_2(v_2)dv_2$.

The term $\bar{x}_1(s)-g(s)\bar{p}_1(s)$ in (4.3) may be interpreted as a *cross-subsidy* relative to the "naive construction" discussed in Section 3. Indeed, suppose that seller type s was not "pooled" with any other types in its trading. In order for type s to trade with all buyer types greater than $g(s)$ with probability one, but to trade with all buyer types less than $g(s)$ with probability zero, one would naively expect the seller to charge a take-it-or-leave-it price of $g(s)$. Since the probability with which type s trades is given by $\bar{p}_1(s)$, the expected revenues without pooling are given by $g(s)\bar{p}_1(s)$. However, in the mechanism $\{p, x\}$, the actual expected revenues of type s are given by $\bar{x}_1(s)$. The difference, $\bar{x}_1(s)-g(s)\bar{p}_1(s)$, may consequently be viewed as an implicit cross-subsidy received by type s from pooling with other types.

Now that we have an interpretation of the $B_1(\cdot)$ function, we will focus on the properties of $B_1(\cdot)$ functions associated with ex ante efficient mechanisms. It is illuminating to consider the Chatterjee-Samuelson mechanism (as defined in (3.1)). Some direct computations show that, in this example, $B_1(s)=(s/6)(3/4-s)^2$, for $s\in[0, 3/4]$, and $B_1(s)\equiv 0$, for $s\in(3/4, 1]$; this is depicted in Figure 4. Observe that $B_1(\cdot)$ takes on the value zero at 3/4, is constant on the interval [3/4, 1], is strictly quasi-concave on [0, 3/4], and is maximized at 1/4; recall that these were the properties which were essential for making equation (4.1) provide an unambiguous pairing of [0, 1/4) with [1/4, 3/4). Also observe that the Chatterjee-Samuelson boundary, $g(s)=s+1/4$, is continuous and strictly increasing on [0, 3/4), and crosses the monopoly boundary, $g^*(s)=s/2+1/2$, precisely once; these are technical results which are used in the ensuing proofs.[20]

Let us refer to any ICBM satisfying the condition $B_1(1)=B_2(0)=0$ as a *balanced mechanism.* In view of equations (4.3) and (4.4), a mechanism is balanced if and only

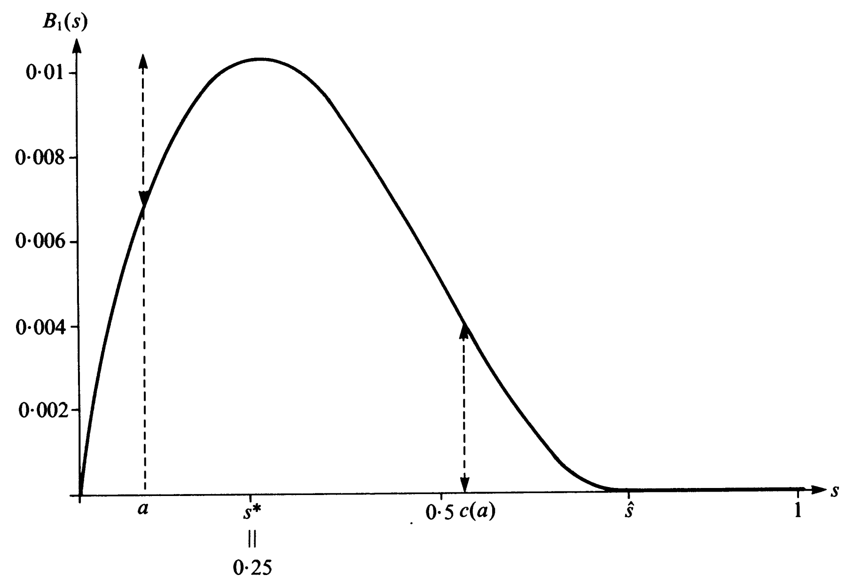


FIGURE 4

The aggregate cross-subsidy function, $B_1(s)$, for the Chatterjee-Samuelson mechanism.

20. Property (ii) of Lemma 4.2 is used both in establishing property (iii) and in showing, in the proof of the main theorem in Appendix B, that the Jacobian of the system of equations (B.9) is non-zero. Properties (i) and (iii) are used in Section 5 to argue that ex ante efficient mechanisms meet the requirements of Theorem 5.2 and, hence, induce two-price mechanisms provided that inequality (5.7) is satisfied.

if cross-subsidies are conserved across all types of each player.[21] Let us also generally define the *monopoly boundary* by $g^*(s) \equiv \arg\max_\phi \pi(\phi, s)$. The aforementioned properties of the Chatterjee-Samuelson mechanism generalize to all distributions satisfying the monotone-hazard-rate assumption, and to all ex ante efficient mechanisms (other than the monopoly mechanism):[22]

Lemma 4.2. *Suppose that $F_1(s)/f_1(s)$ and $[F_2(b)-1]/f_2(b)$ are strictly increasing functions and suppose that $\{p, x\}$ is any ex ante efficient mechanism (other than the monopoly mechanism). Then:*

(i) *$\{p, x\}$ is a balanced 0–1 mechanism with a strictly increasing and continuous boundary $g(\cdot)$;*
(ii) *there exists $\tilde{s} \in (0, \hat{s})$ such that $g(s) < g^*(s)$ for every $s \in [0, \tilde{s})$ and $g(s) > g^*(s)$ for every $s \in (\tilde{s}, \hat{s})$; and*
(iii) *$B_1(\cdot)$ is strictly quasi-concave on $[0, \hat{s}]$, with maximum at $s^* \in (0, \hat{s})$.*

Proof. See Appendix A. ||

The intuition for property (i), the balancedness of ex ante efficient ICBM's, is that the participation constraint requires interim individual rationality; at an optimum, the constraint becomes binding, rendering the mechanism balanced (see footnote 21). The intuition for property (ii) is that, for any ex ante efficient mechanism (other than the monopoly mechanism), the boundary $g(\cdot)$ reaches 1 at an $\hat{s} < 1$, as seller-buyer pairs $(s, b) \approx (1, 1)$ contribute few gains from trade relative to the incentive cost of permitting them to trade. However, the monopoly boundary $g^*(\cdot)$ reaches 1 at $\hat{s} = 1$.

The intuition for property (iii), the quasi-concavity of $B_1(\cdot)$, is quite subtle. Suppose that $s' < s$ and $g^*(s') = g(s)$. Then, in order to deter s' from mimicking s, it must be the case that s receives a smaller cross-subsidy than does s'. Conversely, suppose that $s' > s$ and $g^*(s') = g(s)$. Then, in order to deter s' from mimicking s, it must be the case that s' receives a larger cross-subsidy than does s. Using property (ii), this shows that the cross-subsidy is decreasing in s until the point where $g(s) = g^*(s)$, and is increasing in s thereafter. At the same time, seller types above $\hat{s}$ never trade in the mechanism with boundary $g(\cdot)$, and so earn zero expected utility. In order to deter seller types slightly greater than $\hat{s}$ from mimicking seller types slightly less than $\hat{s}$, it must be the case that types near $\hat{s}$ receive negative cross-subsidies. Combining this with the fact that cross-subsidies are first decreasing and then increasing, and the fact that the aggregate cross-subsidy equals zero, this implies the existence of a unique $s^* \in (0, \hat{s})$ such that all types $s \in (0, s^*)$ receive a positive cross-subsidy and all types $s \in (s^*, \hat{s})$ receive a negative cross-subsidy.

21. Balancedness may also be given a somewhat different interpretation. By Theorem 1 of Myerson and Satterthwaite (1983), it is known that $G_1(1) = G_2(0) = U_2(0) + U_1(1)$. Using our equation (4.2), $B_1(1) = -U_2(0)$ and $B_2(0) = -U_1(1)$. Consequently, an equivalent definition of a balanced mechanism is an ICBM in which the weak types of each player are held to their reservation utilities, i.e. $U_1(1) = 0 = U_2(0)$.

22. At many points in the article (such as here), we will be making arguments from the seller's viewpoint, when thoroughly analogous arguments could equally be made from the buyer's viewpoint. We do this purely for expositional convenience. The reader should thus be alerted to the fact that a symmetric version of Lemma 4.2 (ii)–(iii) holds for the buyer. For example, in the Chatterjee-Samuelson mechanism, $B_2(b) = ((1-b)/6)(b-1/4)^2$, which is a strictly quasi-concave function on the interval $[1/4, 1]$, with $B_2(1/4) = 0 = B_2(1)$ and peak $b^* = 3/4$. Additionally, the Chatterjee-Samuelson boundary crosses the "monopsony boundary", $g^{**}(s) = 2s$ for $s \in [0, 1/2]$, once from above.

We must stress here that *most* mechanisms are imbalanced. For example, starting with any ex ante efficient mechanism, a parallel shift of the boundary to the northwest yields a 0–1 mechanism which, while still consistent with incentive compatibility and individual rationality, must now be imbalanced.[23] But balancedness is a natural property for equilibria of games without mediators, since any cross-subsidy received by one player type must be paid by another. In fact, we have:

Lemma 4.3. *Let $\{p, x\}$ be any ICBM induced by a sequential equilibrium of an offer/counteroffer bargaining game in which the discount factor between periods is bounded away from zero. Then $\{p, x\}$ is a balanced mechanism.*

Proof. See Appendix A. ||

The contrapositive of Lemma 4.3 thus establishes that imbalanced mechanisms are not implementable. However, Lemma 4.2 demonstrated that considerations of balancedness do not interfere with the implementation of efficient mechanisms.

Nonetheless, Lemma 4.2 also suggests that we will need to utilize equilibria which involve *pooling into non-convex sets.* The intuition is as follows. Recall that pooling is necessary in order for there to be any cross-subsidization. Now consider any equilibrium which involves any variety of pooling in the seller's initial offer, and let S be the pool of all seller types who make one particular initial offer. Note that if the seller belongs to S, then it would be incentive compatible for her to announce "I belong to pool S" before making her initial offer. Similarly, if the seller belongs to the complement of S, then she can be induced to announce "I do not belong to pool S" before making her initial offer. We will denote such an announcement a *seller split.*[24] If S happens to be an interval of the form $[0, s)$, and so the complement of S is $[s, 1]$, then we will refer to this as a *convex split at s.*

Notice that, after she has announced that she belongs to S but before she makes her initial offer, seller $v_1 \in S$ is still supposed to trade with exactly those buyer types in the interval $(g(v_1), 1]$. Without cross-subsidization from other seller types, this entitles type v_1 to revenues equaling $g(v_1)\bar{p}_1(v_1)$. Since her actual expected revenues are still $\bar{x}_1(v_1)$, and since no cross-subsidization can come from outside pool S, it must be the case that $\int_S [\bar{x}_1(v_1) - g(v_1)\bar{p}_1(v_1)]f_1(v_1)dv_1 = 0$. (Compare with equation (4.3).) Thus, just as there is conservation of cross-subsidies in the aggregate, there must also be conservation of cross-subsidies in each pool. More precisely, for convex revelation,[25] we have the following lemma:[26]

Lemma 4.4. *Suppose $\{p, x\}$ is an ICBM induced by a sequential equilibrium of an offer/counteroffer bargaining game, and suppose that a convex split at s by the seller is possible. Then $B_1(s) = 0$.*

23. Unless, of course, $p(\cdot, \cdot) \equiv 0$.

24. Obviously an entirely analogous definition holds for buyer splits.

25. In the working paper precursor of this article (Ausubel and Deneckere (1988*b*, Theorem 3.6)), we establish a general characterization of non-convex binary splits. Let $0 = s_0 < s_1 < s_2 \cdots < s_n = 1$, and let $I = [0, s_1) \cup [s_2, s_3) \cup \cdots$ and $II = [s_1, s_2) \cup [s_3, s_4) \cup \cdots$ be a binary partition of $[0, 1]$. The following proposition is demonstrated: If the ICBM $\{p, x\}$ is seller-splittable in the binary partition $\{I, II\}$, then $\sum_{i=0}^{n} (-1)^i B_1(s_i) = -\alpha(n) U_2^I(0)$, where $\alpha(n) = a_{n-1}$ if n is even, $\alpha(n) = a_n$ if n is odd, and where $a_k = F_1(s_k) - F_1(s_{k-1}) + F_1(s_{k-2}) - F_1(s_{k-3}) + \cdots \pm F_1(s_1)$.

26. Although the result is stated in terms of the initial mechanism $\{p, x\}$, i.e. relative to the prior distribution functions $F_1(\cdot)$ and $F_2(\cdot)$, it can also be applied repeatedly to the resulting sub-mechanisms (and the respective conditional distributions).

(*Analogously, if* $\{p, x\}$ *is convexly splittable by the buyer at b, then* $B_2(b)=0$.)

Proof. See Appendix A. ||

Lemma 4.2 and Lemma 4.4, combined, explain why we have resorted to equilibria with non-convex revelation structures. Suppose that an efficient mechanism could be replicated by a sequential equilibrium in which the initial offer convexly partitioned seller types into disjoint intervals, the lowest of which was $[0, s)$. Lemma 4.4 showed that aggregate cross-subsidies in the interval $[0, s)$ must be zero, i.e. $B_1(s)=0$. Lemma 4.2 established that if the equilibrium was to replicate an efficient mechanism, then in fact $B_1(s)>0$ for all $s \in (0, \hat{s})$. This yields a contradiction.

5. TWO-PRICE MECHANISMS

In Section 3, by considering the limits of our sequential equilibria as $N \to \infty$ and $\delta \to 1$, we suggested that efficient mechanisms can be interpreted as two-stage trading processes. In a first stage, sellers of different valuations pairwise pool: for every $a \in [0, s^*)$, there is associated a unique $c(a) \in [s^*, \hat{s})$ such that types a and $c(a)$—but nobody else—offers the initial price p_0. More precisely, sellers with valuation $s \in [0, \hat{s})$, by naming the initial price $p_0(s)$, reveal which doubleton $\{a, c(a)\}$ they belong to; sellers with valuations in $[\hat{s}, 1]$ merely pool by making non-serious offers. In the second stage, the doubletons split into singletons: seller $s \in \{a, c(a)\}$ reveals whether $s=a$ or $s=c(a)$ by offering the boundary price $g(a)$ or $g(c(a))$, respectively. Sellers with valuation $s \geqq \hat{s}$ continue to make non-serious offers. The two stages occur in virtual time (i.e. without discounting between periods), and buyers merely optimize given their knowledge of the offer structure and their conviction that prices will not be reduced beyond the second stage. More precisely, in the first stage all buyer types with valuations in $[\beta(s), 1]$ purchase at the price $p_0(s)$. Let $\theta(a)$ be the probability that $s=a$, conditional on the event that $s \in \{a, c(a)\}$. With probability $\theta(a)$, the price in the second stage falls from $p_0(a)$ to $g(a)$; in this event all buyer types in $[g(a), \beta(a))$ purchase. With probability $(1-\theta(a))$, the price in the second stage rises to $g(c(a))$; in this event, only buyer types in $[g(c(a)), \beta(a))$ purchase. In either event, no further trade ever takes place.

Our game plan is first to investigate the limiting functions $c(\cdot)$, $p_0(\cdot)$, $\beta(\cdot)$ and $\theta(\cdot)$ needed to replicate efficient mechanisms. Afterward (and mostly in Appendix B), we then indicate how these functions may be perturbed to generate the N sub-intervals and functions $\hat{p}_0(\cdot)$, $\hat{\beta}(\cdot)$ and $\hat{\theta}(\cdot)$ which are required to complete the specification of the equilibrium from Section 3.

Several conditions must be satisfied in order for the (limiting) two-stage revelation to be consistent. First, since the seller types a and $c(a)$ are indistinguishable to the buyer at the outset:

$$p_0(c(a))=p_0(a) \quad \text{and} \quad \beta(c(a))=\beta(a) \quad \text{for all } a \in [0, s^*). \tag{5.1}$$

Second, for sales to be non-negative at both the first and second offer, we need:

$$g(s) \leqq \beta(s) < 1 \quad \text{for all } s \in [0, \hat{s}). \tag{5.2}$$

Third, the conditional probability $\theta(\cdot)$ is determined by the distribution function $F_1(\cdot)$ and the pairing function $c(\cdot)$:[27]

$$\theta(a)=\frac{f_1(a)}{f_1(a)+f_1(c(a)) \cdot (dc/da)(a)}. \tag{5.3}$$

27. Note that (5.3) is the limiting form of (3.2).

Finally, if some buyer types purchase in the initial period while others purchase in the next period, and $\delta \to 1$, the initial offer must equal the conditional expectation of the next offer:

$$p_0(a) = \theta(a)g(a) + [1-\theta(a)]g(c(a)) \quad \text{for all } a \in [0, s^*). \tag{5.4}$$

The probability of trade and expected payment corresponding to this two-stage trading process are thus given by:

$$p(s,b) = \begin{cases} 1 \\ 1, \\ 0 \end{cases} \quad x(s,b) = \begin{cases} p_0(s) & \text{if } \beta(s) \leqq b \leqq 1 \\ g(s) & \text{if } g(s) \leqq b < \beta(s) \\ 0 & \text{otherwise.} \end{cases} \tag{5.5}$$

If $c(\cdot)$, $p_0(\cdot)$, $\beta(\cdot)$ and $\theta(\cdot)$ satisfy these consistency conditions (in association with the boundary $g(\cdot)$ of a balanced 0–1 mechanism), we will call this a two-price mechanism. More precisely:

Definition 5.1. The quintuplet $\{g(\cdot), c(\cdot), p_0(\cdot), \beta(\cdot), \theta(\cdot)\}$ will be called a *two-price mechanism* if:

(i) $c:[0, s^*) \to [s^*, \hat{s})$ is a continuously-differentiable function with $dc/ds > 0$ on $(0, s^*)$;
(ii) (5.1), (5.2), (5.3) and (5.4) are satisfied; and
(iii) $\{p, x\}$ determined by (5.5) is a balanced ICBM.

Conversely, suppose we are given a 0–1 mechanism with boundary $g(\cdot)$. We will say that $g(\cdot)$ *induces a two-price mechanism* if there exists $s^* \in (0, \hat{s})$ and functions $c(\cdot)$, $p_0(\cdot)$, $\beta(\cdot)$ and $\theta(\cdot)$ such that $\{g(\cdot), c(\cdot), p_0(\cdot), \beta(\cdot), \theta(\cdot)\}$ satisfies (i), (ii) and (iii) above.

There is no a priori reason to believe that every balanced 0–1 mechanism would induce a two-price mechanism (and, in fact, most do not). Recall that, in a two-price mechanism, all seller types in the interval $[0, s^*)$ receive an implicit cross-subsidy and all seller types in $(s^*, \hat{s})$ pay an implicit cross-subsidy: types a and $c(a)$ pool and make sales at $p_0(a)$ which satisfies $g(a) \leqq p_0(a) \leqq g(c(a))$. Consequently (in view of Lemma 4.1), in order for a balanced 0–1 mechanism to induce a two-price mechanism, it must be that the associated $B_1(\cdot)$ function is upward-sloping on the sub-interval $[0, s^*)$ and downward sloping on the sub-interval $(s^*, \hat{s})$. This insight leads to the following remarkable theorem, which links the equilibrium construction of Section 3 with the mechanism results of Section 4:

Theorem 5.2. *Let $g(\cdot)$ be continuous and strictly monotone on $[0, \hat{s})$ and suppose that $g(\cdot)$ induces a two-price mechanism with pairing $c(\cdot)$. Then $B_1(\cdot)$ is strictly quasi-concave. Furthermore, letting $s^* = \arg\max_s B_1(s)$:*

$$B_1(a) + B_1(c(a)) = B_1(s^*) \quad \textit{for all } a \in [0, s^*). \tag{5.6}$$

Conversely, let $g(\cdot)$ be a strictly increasing continuous boundary associated with a balanced 0–1 mechanism having a strictly quasi-concave $B_1(\cdot)$ on $[0, \hat{s}]$. Let $c(\cdot)$ be the strictly increasing function from $[0, s^*)$ to $[s^*, \hat{s})$ defined implicitly by (5.6). *Then $g(\cdot)$ induces a two-price mechanism with pairing $c(\cdot)$, provided that the $\beta(\cdot)$ it implies is consistent with*:

$$g(s) \leqq \beta(s) \quad \textit{for all } s \in [s^*, \hat{s}). \tag{5.7}$$

Proof. Note from (5.5), for a 0–1 mechanism, that $\bar{p}_1(s) = 1 - F_2(g(s))$ and that

$$\begin{aligned} \bar{x}_1(s) &= [1 - F_2(\beta(s))]p_0(s) + [F_2(\beta(s)) - F_2(g(s))]g(s) \\ &= [1 - F_2(\beta(s))][p_0(s) - g(s)] + g(s)\bar{p}_1(s). \end{aligned}$$

Hence, using (4.5) we have for every s:

$$[1-F_2(\beta(s))][p_0(s)-g(s)]=B_1'(s)/f_1(s). \tag{5.8}$$

Consequently, using (5.8) and (5.1):

$$[p_0(a)-g(a)]/[p_0(a)-g(c(a))]$$
$$=[B_1'(a)/f_1(a)]/[B_1'(c(a))/f_1(c(a))] \quad \text{for all } a\in[0,s^*). \tag{5.9}$$

In order for (5.4) to be satisfied, we must have:

$$1-1/\theta(a)=[p_0(a)-g(a)]/[p_0(a)-g(c(a))] \quad \text{for all } a\in[0,s^*). \tag{5.10}$$

Combining (5.9) and (5.10), this implies:

$$\theta(a)=-[B_1'(c(a))/f_1(c(a))]/[B_1'(a)/f_1(a)-B_1'(c(a))/f_1(c(a))],$$
$$\text{for all } a\in[0,s^*). \tag{5.11}$$

Since the density $f_1(s)$ is positive, (5.3) and (5.11) imply that (5.6) holds. Moreover, (5.3), (5.4) and Definition 5.1(i) imply that $g(a)<p_0(a)=p_0(c(a))<g(c)$, for $a\in(0,s^*)$. Equation (5.8) and the second inequality in (5.2) then yield that $B_1'(a)>0$ and $B_1'(c(a))<0$ for all $a\in(0,s^*)$, establishing quasi-concavity and proving the first part of the theorem.

Suppose conversely that $c(\cdot)$ satisfies (5.6). Since $g(\cdot)$ is continuous, $B_1(\cdot)$ is differentiable everywhere. Evaluating (5.8) at a and $c(a)$, and subtracting the resulting expressions, yields a unique candidate $\beta(a)=\beta(c(a))$:

$$1-F_2(\beta)=[B_1'(a)/f_1(a)-B_1'(c(a))/f_1(c(a))]/[g(c(a))-g(a)]. \tag{5.12}$$

By the strict quasi-concavity of $B_1(\cdot)$ and the strict monotonicity of $g(\cdot)$, the right-hand side of (5.12) is positive for all $a\in[0,s^*)$, and hence any solution satisfies $\beta<1$. We will now show that if (5.7) (i.e. the remaining half of inequality (5.2)) is satisfied, then we can define functions $p_0(\cdot)$ and $\theta(\cdot)$ such that all the other requirements for a two-price mechanism are met. Since $\beta(s)<1$, then (5.8) evaluated at a (or equivalently, at $c(a)$) yields a solution satisfying $g(a)\leqq p_0(a)\leqq g(c(a))$. $\theta(a)$ is then defined from (5.10), implying that (5.4) holds. From (5.9) and (5.6), it then follows that, at every a in $[0,s^*)$, (5.3) holds. ||

Equation (5.6) establishes that the pairing between a and $c(a)$ can be read directly from the $B_1(\cdot)$ function, as illustrated in Figure 4.

Theorem 5.2 reduces the complicated problem of showing that a balanced 0–1 mechanism is a two-price mechanism to the vastly simpler problem of demonstrating that, first, the associated $B_1(\cdot)$ function is strictly quasi-concave and, second, the additional requirement of (5.7) is met. Already, in Lemma 4.2, we have shown that under the distributional assumption that $F_1(s)/f_1(s)$ and $[F_2(b)-1]/f_2(b)$ are increasing, any ex ante efficient mechanism other than the monopoly mechanism has a strictly quasi-concave $B_1(\cdot)$. Unfortunately, (5.7) is a troublesome constraint. If (5.7) is not satisfied, then for some $c\in[s^*,\hat{s})$, the initial sales, $1-F_2(\beta(c))$, exceed the sales prescribed by the mechanism, $1-F_2(g(c))$. In this case, rather than making further sales, c would have to buy back the quantity $F_2(g(c))-F_2(\beta(c))$ at the price $g(c)$, and make no further sales thereafter. Obviously, such behaviour cannot be sustained in equilibrium.

To understand why inequality (5.7) may fail, recall that (4.6) provides the cross-subsidy paid by type $s\in(s^*,\hat{s})$. Hence, the magnitude of the cross-subsidy is entirely prespecified by the function $B_1(\cdot)$. Now observe that $B_1(\cdot)$ also predetermines p_0, since

p_0 is a weighted average of the two boundary prices (see (5.4)), where the weight θ is uniquely determined by (5.6) and (5.3). From (5.8), we then see that (5.7) will be violated at some $s \in (s^*, \hat{s})$ if, in some sense, the cross-subsidy required of s is "too large".

A useful way to rewrite constraint (5.7) is:

$$\bar{x}_1(s)/\bar{p}_1(s) \geqq p_0(s) \quad \text{for all } s \in [s^*, \hat{s}) \text{ s.t. } \theta(s) \neq 0. \tag{5.13}$$

Indeed, in a 0–1 mechanism, $\bar{x}_1(s)/\bar{p}_1(s)$ represents the average price paid to seller s. In a two-price mechanism, $p_0(s)$ and $g(s)$ are the only prices ever paid, and for $s \in [s^*, \hat{s})$ s.t. $\theta(s) \neq 0$, $g(s) > p_0(s)$. Hence, (5.13) is satisfied if and only if sales are non-negative at $g(s)$.

For any given mechanism it is, of course, possible to directly verify whether (5.7)—or, equivalently, (5.13)—holds. However, we would like to have sufficient conditions for efficient mechanisms to be representable as two-price mechanisms, expressed directly in terms of the primitives of the model (i.e. the distribution functions $F_1(\cdot)$ and $F_2(\cdot)$). In addition, the reader should realize that in our proof of implementation, the limiting two-price mechanism is approached with discrete approximations. In order to derive such approximations, at least one of the two inequalities (5.7) and (5.13) will need to be satisfied with slack. Observe that (5.7) cannot have slack at $\hat{s}$, since $g(\hat{s}) = \beta(\hat{s}) = 1$. Nor can constraint (5.13) generally have slack at s^*.[28] However, it is often possible for (5.7) to have slack at s^*, and for (5.13) to have slack at $\hat{s}$:

Lemma 5.3. *Suppose that $F_1(s)/f_1(s)$ and $[F_2(b)-1]/f_2(b)$ are strictly increasing functions. Then there exists $\lambda_s \in (0,1)$ such that, for every $\lambda \in (\lambda_s, 1)$, there exists $\varepsilon^\lambda > 0$ such that the ex ante efficient mechanism with weight λ on the seller induces a boundary g^λ satisfying at least one of the constraints* (5.7) *and* (5.13) *with slack ε^λ.*

Proof. See Appendix A. ||

Subject to these distributional assumptions, Theorem 5.2 and Lemma 5.3 jointly establish that ex ante efficient mechanisms (other than the monopoly mechanism) induce two-price seller mechanisms, for weights $\lambda \in [\lambda_s, 1]$. Analogously, ex ante efficient mechanisms (other than the monopsony mechanism) with weight $\lambda \in [0, \lambda_b]$ induce two-price buyer mechanisms, for some $\lambda_b \in (0,1)$. Loosely speaking, then, these theorems show the existence of "limiting equilibria" (when $z=0$ and $N=\infty$) that implement a portion of the Pareto frontier.

However, the construction of two-price mechanisms is not quite sufficient to conclude this enterprise, as we require sequential equilibria of *infinite-horizon, discrete-time* games. The proof of the main theorem proceeds in three steps (see Appendix B). First, we construct a sequence of discrete two-price mechanisms which approach the limiting two-price mechanism. The definition of a discrete two-price mechanism parallels Definition 5.1 but, instead, $c: [0, s^*) \to [s^*, \hat{s})$ is now given by a step function. Thus, there exists a grid $s^* = c_0 < c_1 < \cdots < c_{N-1} < c_N = \hat{s}$ of $(N+1)$ seller types partitioning the interval $[s^*, \hat{s}]$, and a corresponding grid $0 = a_0 < a_1 < \cdots < a_{N-1} < a_N = s^*$ of seller types in $[0, s^*]$ given by (B.2), such that seller types belonging to the paired intervals $[a_k, a_{k+1})$ and $[c_k, c_{k+1})$ have the common boundary $g(a_k)$ and $g(c_k)$, respectively. (The conditional probability that $s \in [a_k, a_{k+1})$ is then given by (3.2).) Second, we approximate the discrete

28. From (5.8), $p_0(s^*) = g(s^*)$, and from (5.13), $\bar{x}_1(s^*) - g(s^*)\bar{p}_1(s^*) = 0$; combining these equations implies that $\bar{x}_1(s^*)/\bar{p}_1(s^*) = p_0(s^*)$.

two-price mechanisms in turn with mechanisms that have discounting between the initial offer $p_0(\cdot)$ and the subsequent offers. Equation (5.4) is then replaced by:

$$\beta(a_k)-p_0(a_k)=\delta\{\beta(a_k)-\theta(a_k)g(a_k)-[1-\theta(a_k)]g(c_k)\},$$

since the buyer with valuation $\beta(a_k)$ should be indifferent between the initial offer and following (expected) offer. Third, we replace the price paths assigned to $[a_0, a_1)$: rather than offering the price $p_0(a_0)$ followed by the price $g(a_0)$, we allow those sellers to offer an initial price followed by an exponentially-descending price path in future periods. The corresponding menu of price paths is then supported by stationary sequential equilibria of the type discussed in Section 3. This completes the proof of the main theorem.

While Theorem 3.1 guarantees the implementability of a portion of the Pareto frontier, namely, the ex ante efficient mechanisms with weights $\lambda\in[0, \lambda_b]\cup[\lambda_s, 1]$, it provides no information on how large this portion may be. We will now show, at least for some interesting classes of distributions, that $\lambda_s\leqq\lambda_b$ so that the entire Pareto frontier can be implemented.

Example 5.4. Consider the Chatterjee–Samuelson mechanism (3.1) associated with equal weighting ($\lambda=1/2$) and uniform distributions. The pairing $c(a)$ is derived from (5.6), using $B_1(s)=(s/6)(3/4-s)^2$. It can be demonstrated that dc/da is a strictly decreasing function on $(0, 1/4)$, with range $(1, \infty)$. In addition, $\beta(a)=(1/2)[1+a+c(a)]$ (see Figure 3). Constraint (5.7) holds everywhere on $[s^*, \hat{s})$, but $\beta(\cdot)$ and $g(\cdot)$ are tangent at $\hat{s}$, implying that both (5.7) and (5.13) hold with equality as s approaches $\hat{s}$. As we increase λ from 1/2, constraint (5.13) is monotonically relaxed. Meanwhile, (5.13) is violated at $\hat{s}$ for every $\lambda<1/2$. A symmetric argument establishes that λ_b also equals 1/2. We conclude that, in the double-uniform case, the seller-offer game spans the half of the efficiency frontier favouring the seller, while the buyer-offer game spans the remaining half, which favours the buyer.

Example 5.5. Let $F_1(s)=s^\alpha$ and $F_2(b)=1-[1-b]^\gamma$, for $\alpha, \gamma>0$. The set of ex ante efficient boundaries is then parametrically defined by $g(s)=\rho+(1-\rho)s/\hat{s}$, where $\hat{s}=[\alpha+\rho(1+\gamma)]/(1+\alpha)$ and ρ $(0\leqq\rho\leqq 1/(\alpha+\gamma))$ is a parameter reflecting the relative weight to the seller. In particular, if $\rho=0$, we obtain the monopsony mechanism, if $\rho=1/(\alpha+\gamma)$ we obtain the monopoly mechanism, and if $\rho=(\alpha-D^{1/2})/(\alpha-\gamma)$ with $D=\alpha\gamma[(1+\alpha)/(1+\gamma)]$ we obtain equal weighting. Some tedious algebra establishes that $B_1(s)=[(1-\rho)/\hat{s}]^\gamma[1/(1+\gamma)-\rho/\hat{s}][\hat{s}-s]^{1+\gamma}s^\alpha$, and that $s^*=[\alpha/(1+\alpha+\gamma)]\hat{s}$. While $F_1(\cdot)$ and $F_2(\cdot)$ do not literally satisfy the distributional assumptions of this article—density $f_1(\cdot)$ is either infinite or zero at $s=0$, and $f_2(\cdot)$ is either infinite or zero at $b=1$—we show in the working paper precursor of this article (Ausubel and Deneckere (1988*b*)) that the proofs go through under the weakened assumption that the density functions are defined only on the interior of the support, $\lim_{s\downarrow 0} f_1(s)>0$ and $\lim_{b\uparrow 1} f_2(b)>0$. Consequently, we restrict attention to the parameter values $0<\alpha$, $\gamma\leqq 1$. Numerical computations indicate that (5.13) is satisfied for the entire rectangle $0<\alpha$, $\gamma\leqq 1$, $\lambda_s\leqq 1/2$ and $\lambda_b\geqq 1/2$.

6. CONCLUSION

In earlier work (Ausubel and Deneckere (1989*a*, *b*)), we established two qualitative propositions concerning sequential bargaining with *one-sided* incomplete information.

First, two extensive forms (the seller-offer and buyer-offer games) are sufficient to implement the entire (ex ante) Pareto frontier. Second, the ability to make offers confers bargaining power.[29]

In the current article, we demonstrated similar results for two-sided incomplete information. For fairly general distributions, one segment of the efficient frontier is implementable in the seller-offer game and another segment is implementable in the buyer-offer game. Often, the union of these two segments equals the entire Pareto frontier (see, again, Figure 1). At the same time, the segment we construct from seller-offer equilibria necessarily includes those static mechanisms most favourable to the seller, while usually excluding those mechanisms most favourable to the buyer (and, analogously, for the buyer-offer game). This again draws a connection between bargaining strength and the exclusive ability to make offers.

In one-sided incomplete information, some equilibria of the seller-offer and buyer-offer games can be "embedded" in extensive forms (e.g. alternating offer) which permit both parties to make offers (Fudenberg, Levine and Tirole (1985), Ausubel and Deneckere (1989*b*)). It may also be possible to do this in the case of two-sided incomplete information. We plan to pursue this issue in future work.

APPENDIX A

Proof of Lemma 4.2. (i) By Theorem 4 of Williams (1987), $b=g(s)$ satisfies the equation:

$$b-\sigma[1-F_2(b)]/f_2(b)=s+\tau[F_1(s)/f_1(s)]. \tag{A.1}$$

for some $\sigma, \tau \in [0,1]$ with $\sigma \neq 1$ and $\tau \neq 0$. Since $F_1(s)/f_1(s)$ and $[F_2(b)-1]/f_2(b)$ are continuous as well as strictly increasing functions, $g(\cdot)$ is also continuous and strictly increasing. Balancedness follows from Theorem 3 of Williams (1987).

(ii) The monopoly boundary $g^*(\cdot)$ must satisfy $g^*(s)=\arg\max_\phi \pi(\phi, s)$, or:

$$g^*(s)-s-[1-F_2(g^*(s))]/f_2(g^*(s))=0. \tag{A.2}$$

Subtracting (A.2) from (A.1) yields:

$$(1-\sigma)[1-F_2(g^*(s))]/f_2(g^*(s))=\tau[F_1(s)/f_1(s)],$$

at any s such that $b=g(s)=g^*(s)$. Since $g^*(s)$ is strictly increasing in s, the left-hand side of this equation is strictly decreasing in s, while the right-hand side is strictly increasing. Consequently, there is at most one value $\tilde{s}$ such that $g(\tilde{s})=g^*(\tilde{s})$. At the same time, at $s=0$, the left-hand side strictly exceeds the right-hand side, and vice versa for $s=1$. Consequently, the existence of $\tilde{s}\in(0,1)$ is guaranteed. Meanwhile, using the above equations for the boundary, it is straightforward to show that $g(s)<g^*(s)$ for $s<\tilde{s}$, and the reverse for $s>\tilde{s}$.

(iii) Define $\mu_1(s)\equiv B_1'(s)/f_1(s)$, as provided by (4.6). We will show that there exists a unique $s^*\in(0,\hat{s})$ such that $\mu_1(s^*)=0$. Since $g(\cdot)$ is monotone, $\mu_1'(s)$ exists almost everywhere and

$$\mu_1'(s)=g'(s)f_2(g(s))\{g(s)-s-[1-F_2(g(s))]/f_2(g(s))\}.$$

Observe that $\mu_1'(s)>0$ if $s>\tilde{s}$, and $\mu_1'(s)<0$ if $s<\tilde{s}$. Then since $\mu_1(\cdot)$ is increasing on $[\tilde{s},\hat{s}]$, and since $\mu_1(\hat{s})=0$ by definition, it follows that $\mu_1(s)<0$ on $[\tilde{s},\hat{s}]$. On $[0,\tilde{s}]$, on the other hand, μ_1 is strictly decreasing, and hence can have at most one zero. Consequently, $B_1(\cdot)$ can have at most one stationary point in $[0,\hat{s}]$. By definition, $B_1(0)=0$, and by balancedness, $B_1(\hat{s})=0$. Since $B_1(s)>0$ on $[\tilde{s},\hat{s}]$, such a stationary point exists and it is a maximum. ||

Proof of Lemma 4.3. We will argue that $U_2(0)=0$. The argument for $U_1(1)$ proceeds in an entirely analogous fashion.

29. In one-sided incomplete information, if the seller's (buyer's) valuation is commonly known, *all* ICBM's are implementable in the seller- (buyer-) offer game. Modifying the extensive form, by permitting the silent party to make offers, eliminates those ICBM's most unfavourable to the silent party. These propositions are proven for the case of "no gap", i.e. the uninformed party's valuation is contained in the support of the informed party's valuation.

First, since acceptances are ex post IR, (strictly) negative buyer offers are rejected by the seller with probability one. It will follow that $U_2(0)=0$ if we can show that the seller never offers a (strictly) negative price. This will certainly be the case if negative prices have probability one of acceptance. Let Q be the infimum of all seller offers which are rejected with positive probability (the infimum taken over all seller types, all sequential equilibria and all histories). Using an argument entirely analogous to Fudenberg, Levine and Tirole (1985, Lemma 2) we can establish that the lowest buyer's equilibrium utility is bounded above by 1. Using the Lipschitz continuity of utility (with Lipschitz constant of one), it follows that a buyer of type b can earn no more than $(1+b)$; consequently, any offer below -1 has probability one of acceptance. We have just shown that $Q \geqq -1$. Next, we claim that $Q=0$. Suppose instead that $Q<0$, and let $\delta<1$ be the minimal discount factor between periods. Then there exists an offer $q<\delta Q$, made by some seller type in some sequential equilibrium after some history, which is rejected with positive probability. But any buyer rejecting such an offer is irrational, since he cannot hope to receive a future offer below Q, and since any negative offer the buyer might make will be rejected with probability one. Consequently, $Q=0$, and $U_2(0)=0$. ||

Proof of Lemma 4.4. Let $I=[0,s)$ and $II=[s,1]$, so that $F_1^I(v_1)=F_1(v_1)/F_1(s)$ and $F_1^{II}(v_1)=[F_1(v_1)-F_1(s)]/[1-F_1(s)]$ are the conditional probability distributions. Also, let $U_1^I(v_1)$ and $U_2^I(v_2)$ refer to the seller's and the buyer's utility in the mechanism $\{p^I, x^I\}$ (relative to the distributions F_1^I and F_2) which results after the seller has announced that she belongs to $[0,s)$ but has not yet traded. By incentive compatibility of $\{p^I, x^I\}$ and Myerson and Satterthwaite (1983, Theorem 1):

$$U_1^I(s)+U_2^I(0)=\int_0^s\int_0^1 \{[v_2-(1-F_2(v_2))/f_2(v_2)]$$
$$-[v_1+F_1^I(v_1)/f_1^I(v_1)]\}p^I(v_1,v_2)f_2(v_2)f_1^I(v_1)dv_2dv_1. \tag{A.3}$$

Notice that, since the seller has not yet traded, $U_1^I(s)=U_1(s)$. Also, $p^I(v_1,v_2)=p(v_1,v_2)$ on $I\times[0,1]$, and hence (A.3) may be rewritten as:

$$U_1(s)+U_2^I(0)=G_1(s)/F_1(s).$$

Now consider the continuation game after the seller splits and reveals that her valuation belongs to $[0,s)$. As argued in the proof of Lemma 4.3, the players never trade at a negative price; individual rationality thus implies $U_2^I(0)=0$. We conclude that $B_1(s)=0$. An analogous argument holds for buyer splits. ||

Proof of Lemma 5.3. By Lemma 4.2(iii) the boundary $g^\lambda(\cdot)$ associated with a seller weight $\lambda<1$ induces a strictly quasi-concave $B_1^\lambda(\cdot)$. Let $\beta^\lambda(\cdot)$, $p_0^\lambda(\cdot)$, $\theta^\lambda(\cdot)$, $\bar{x}_1^\lambda(\cdot)$, $\bar{p}_1^\lambda(\cdot)$ and $\hat{s}^\lambda$ be associated with $g^\lambda(\cdot)$. The proof requires two observations:

a. As $\lambda\uparrow 1$, the function $\beta^\lambda(\cdot)$ converges uniformly to 1. Hence, for sufficiently small $\varepsilon>0$, there exists $\eta(\varepsilon)>0$ such that $\lim_{\varepsilon\downarrow 0}\eta(\varepsilon)=0$ and $\beta^\lambda(s)\geqq g^\lambda(s)+\varepsilon$ for $s\in[0,\hat{s}^\lambda-\eta(\varepsilon))$ and λ in a neighbourhood of 1.
b. Since $\lim_{b\uparrow 1} f_2(b)>0$, it can be shown that $\lim_{\lambda\uparrow 1}\theta^\lambda(\hat{s}^\lambda)>0$. Consequently, $\lim_{\lambda\uparrow 1} p_0^\lambda(\hat{s}^\lambda)<1$, whereas $\lim_{\lambda\uparrow 1}\bar{x}_1^\lambda(\hat{s}^\lambda)/\bar{p}_1^\lambda(\hat{s}^\lambda)=1$. Again, for sufficiently small $\varepsilon>0$, $\bar{x}_1^\lambda(s)/\bar{p}_1^\lambda(s)\geqq p_0^\lambda(s)+\varepsilon$ for $s\in[\hat{s}^\lambda-\eta(\varepsilon),\hat{s}^\lambda]$, and λ in a neighbourhood of 1.

The existence of $\lambda_s<1$ is now immediate. ||

APPENDIX B

Proof of Theorem 3.1. By Lemma 4.2, Theorem 5.2 and Lemma 5.3, there exists $\lambda_s\in(0,1)$ such that, for every $\lambda\in(\lambda_s,1)$, the ex ante efficient mechanism which places weight λ on the seller induces a (continuous) two-price mechanism, and at least one of constraints (5.7) or (5.13) strictly has slack.

Part I: Construction of a discrete mechanism approximating the two-price mechanism, for $\delta=1$.

For any $N>1$, we set $c_0=s^*$, $c_N=\hat{s}$, and arbitrarily select an (increasing) grid $\{c_k\}_{k=1}^{N-1}$ of $N-1$ seller types on the interval $(s^*,\hat{s})$, with the property that the maximum distance between successive sellers on the grid approaches zero as $N\to\infty$. For convenience, let us define:

$$c_k=[(N-k)s^*+k\hat{s}]/N, \qquad k=0,\ldots,N. \tag{B.1}$$

Given $\{c_k\}_{k=0}^N$, we define a second grid $\{a_k\}_{k=0}^N$ on the interval $[0,s^*]$ by:

$$B_1(a_k)+B_1(c_k)=B_1(s^*), \qquad k=0,\ldots,N. \tag{B.2}$$

Observe that $a_0=0$ and $a_N=c_0=s^*$. We will now construct a discrete 0-1 mechanism with associated $\tilde{B}_1(\cdot)$, boundary $\tilde{g}(\cdot)$ and utility function $\tilde{U}_1(\cdot)$, having the property that the utilities from the original mechanism are preserved for seller type $\{c_k\}_{k=0}^N$, i.e. $\tilde{U}_1(c_k)=U_1(c_k)$ for $k=0,\ldots,N$. Since $U_1(s)=\int_s^{\hat{s}}[1-F_2(g(v_1))]dv_1$, this requires:

$$1-F_2(\tilde{g}(s))=\left\{\int_{c_k}^{c_{k+1}}[1-F_2(g(v_1))]dv_1\right\}\Big/(c_{k+1}-c_k),$$

$$\text{for } s\in[c_k,c_{k+1}) \text{ and } k=0,\ldots,N-1, \qquad \text{(B.3)}$$

and $\tilde{g}(s)=1$ for $s\in[\hat{s},1]$. Observe, by (4.6), that $\tilde{B}_1'(s)/f_1(s)=\tilde{x}_1(c_k)-[1-F_2(\tilde{g}(c_k))]\tilde{g}(c_k)$, for all $s\in[c_k,c_{k+1})$. Consequently, $\tilde{B}_1(c_{k+1})-\tilde{B}_1(c_k)=\int_{c_k}^{c_{k+1}}\tilde{B}_1'(s)ds$ is completely determined by (B.1) and (B.3), for $k=0,\ldots,N-1$.

We will now determine values for $\tilde{g}(s)$, $s\in[0,s^*)$, so that $\tilde{B}_1(a_{k+1})-\tilde{B}_1(a_k)=\tilde{B}_1(c_k)-\tilde{B}_1(c_{k+1})$ for $k=0,\ldots,N-1$ and so that $\tilde{g}(\cdot)$ is constant on each $[a_k,a_{k+1})$. Since $\tilde{B}_1'(s)/f_1(s)=\tilde{x}_1(a_k)-[1-F_2(\tilde{g}(a_k))]\tilde{g}(a_k)$, for all $s\in[a_k,a_{k+1})$, we need:

$$\begin{aligned}\tilde{x}_1(a_k)-[1-F_2(\tilde{g}(a_k))]\tilde{g}(a_k)&=[\tilde{B}_1(a_{k+1})-\tilde{B}_1(a_k)]/[F_1(a_{k+1})-F_1(a_k)]\\&=-[\tilde{B}_1(c_{k+1})-\tilde{B}_1(c_k)]/[F_1(a_{k+1})-F_1(a_k)]\equiv d_k.\end{aligned} \qquad \text{(B.4)}$$

Incentive compatibility of a mechanism necessitates that $d\bar{x}_1(s)=sd\bar{p}_1(s)$ (see Myerson and Satterthwaite (1983) and Ausubel and Deneckere (1989*b*, Theorem 1)), implying:

$$\begin{aligned}\tilde{x}_1(a_k)-\tilde{x}_1(a_{k+1})&=a_{k+1}[\tilde{p}_1(a_k)-\tilde{p}_1(a_{k+1})]\\&=a_{k+1}\{[1-F_2(\tilde{g}(a_k))]-[1-F_2(\tilde{g}(a_{k+1}))]\}.\end{aligned} \qquad \text{(B.5)}$$

First-differencing (B.4) and substituting (B.5) into the resulting equation yields:

$$[\tilde{g}(a_k)-a_{k+1}][1-F_2(\tilde{g}(a_k))]-[\tilde{g}(a_{k+1})-a_{k+1}][1-F_2(\tilde{g}(a_{k+1}))]=d_{k+1}-d_k \quad \text{for } k=0,\ldots,N-1. \qquad \text{(B.6)}$$

Let $\pi^*(s)=\max_\phi \pi(\phi,s)=\pi(g^*(s),s)$. Choose small positive Δ. Observe that for sufficiently large N_Δ, $|d_{k+1}-d_k|<\Delta$ for all $N>N_\Delta$ and $k=0,\ldots,N-1$. Suppose that, also, $\inf_k|\pi(\tilde{g}(a_{k+1}),a_{k+1})-\pi^*(a_{k+1})|>\Delta$. Then, equation (B.6) has a solution, $\tilde{g}(a_k)\in[0,g^*(a_{k+1})]$, for $k=0,\ldots,N-1$. However, observe that as $N\to\infty$, the solution to the difference equation (B.6) converges uniformly to the solution of the differential equation:

$$(d/ds)\{[\tilde{g}(s)-s][1-F_2(\tilde{g}(s))]\}=(d/ds)\{B_1'(s)/f_1(s)\} \quad \text{for } s\in[0,s^*]. \qquad \text{(B.7)}$$

Note that (B.7) is uniquely solved by $\tilde{g}(\cdot)\equiv g(\cdot)$ and that $\inf_{s\in[0,s^*]}|\pi(g(s),s)-\pi^*(s)|\equiv\Delta'>0$. Consequently, there exists $\bar{N}>N_{\Delta'/2}>0$ such that, for all $N\geqq\bar{N}$ and $k=0,\ldots,N-1$, the iterative solution to (B.6) satisfies $\inf_k|\pi(\tilde{g}(a_{k+1}),a_{k+1})-\pi^*(a_{k+1})|>\Delta'/2$. Thus, for all $N\geqq\bar{N}$, we have completely defined a discrete boundary, $\tilde{g}(\cdot)$, approximating $g(\cdot)$. Since $\tilde{B}_1(0)=0$, we have also assured $\tilde{B}_1(a_k)+\tilde{B}_1(c_k)=\tilde{B}_1(s^*)$ for all $k=0,\ldots,N$.

In order to argue that the mechanism with boundary $\tilde{g}(\cdot)$ has the two-price interpretation, all that remains to be shown is that $\tilde{g}(c_k)\leqq\tilde{\beta}(c_k)\leqq1$ for $k=0,\ldots,N-1$. The second inequality holds strictly by Theorem 5.2, since $\tilde{B}_1(\cdot)$ was constructed to be strictly quasi-concave with peak at s^*. We will now establish a result somewhat stronger than the first inequality: there exists $\tilde{N}>\bar{N}$ such that for every $N\geqq\tilde{N}$ and for every $k=0,\ldots,N-1$, at least one of $\tilde{\beta}(c_k)>\tilde{g}(c_k)+\varepsilon/2$ and $\tilde{x}_1(c_k)/\tilde{p}_1(c_k)>\tilde{p}_0(c_k)+\varepsilon/2$ holds. We will demonstrate this fact using Lemma 5.3 and the uniform convergence of $\tilde{\beta}(\cdot)$, $\tilde{g}(\cdot)$, $\tilde{x}_1(\cdot)/\tilde{p}_1(\cdot)$ and $\tilde{p}_0(\cdot)$ to $\beta(\cdot)$, $g(\cdot)$, $\bar{x}_1(\cdot)/\bar{p}_1(\cdot)$ and $p_0(\cdot)$. Observe that $\tilde{g}\to g$ uniformly, as $N\to\infty$, because the grid width approaches zero. Define

$$\tilde{\theta}(s)=[F_1(a_{k+1})-F_1(a_k)]/\{[F_1(a_{k+1})-F_1(a_k)]+[F_1(c_{k+1})-F_1(c_k)]\}$$

and

$$\tilde{p}_0(s)\equiv\tilde{\theta}(a_k)\tilde{g}(a_k)+[1-\tilde{\theta}(c_k)]\tilde{g}(c_k),$$

for $s\in[a_k,a_{k+1})\cup[c_k,c_{k+1})$. Note that $\tilde{\theta}\to\theta$ uniformly and so $\tilde{p}_0\to p_0$ uniformly. Now

$$\tilde{x}_1(a_k)=[1-F_2(\tilde{\beta}(a_k))]\tilde{p}_0(a_k)+[F_2(\tilde{\beta}(a_k))-F_2(\tilde{g}(a_k))]\tilde{g}(a_k),$$

and similarly for $\tilde{x}_1(c_k)$. Recall that $\tilde{\beta}(a_k)=\tilde{\beta}(c_k)$ and $\tilde{p}_0(a_k)=\tilde{p}_0(c_k)$; subtracting yields:

$$F_2(\tilde{\beta}(c_k))=\frac{\tilde{x}_1(c_k)-\tilde{x}_1(a_k)+\tilde{g}(c_k)F_2(\tilde{g}(c_k))-\tilde{g}(a_k)F_2(\tilde{g}(a_k))}{\tilde{g}(c_k)-\tilde{g}(a_k)} \qquad \text{(B.8)}$$

and an analogous expression for $F_2(\beta(\cdot))$. Since $\tilde{g}\to g$ uniformly and $g(c)-g(a)$ is bounded away from zero (for all paired a and c), (B.8) shows that $F_2(\tilde{\beta}(\cdot))\to F_2(\beta(\cdot))$ uniformly and hence $\tilde{\beta}\to\beta$ uniformly.

It remains to be demonstrated that $\tilde{x}_1/\tilde{p}_1\to\bar{x}_1/\bar{p}_1$ uniformly. This convergence can be shown algebraically, but it is more informative to argue it graphically, via Figure 5. Begin with the original mechanism $p(\cdot)$. Since $U_1(s)=\int_s^{\hat{s}}\bar{p}_1(v_1)dv_1$ for $s\in[0,\hat{s}]$, and since by hypothesis $g(\cdot)$ is continuous, it should be observed that $U_1(\cdot)$ is C^1 and convex, with slope $-\bar{p}_1(s)$ at s. Let T_s denote the tangent line through s. Since $\bar{x}_1(s)=U_1(s)+s\bar{p}_1(s)$, it follows that $\bar{x}_1(s)$ is the U_1-intercept of T_s, and hence $\bar{x}_1(s)/\bar{p}_1(s)$ is the v_1-intercept of T_s. Now consider the discrete mechanism with boundary $\tilde{g}(\cdot)$. By construction, $\tilde{U}_1(c_k)=U_1(c_k)$ for $k=0,\ldots,N-1$. Let L_k denote the line through $(c_k, \tilde{U}_1(c_k))$ and $(c_{k+1}, \tilde{U}_1(c_{k+1}))$. Analogous to the above, the slope of L_k equals $-\tilde{p}_1(c_k)$ and $\tilde{x}_1(c_k)/\tilde{p}_1(c_k)$ is the v_1-intercept of L_k. Since L_k is also the secant line of U_1 at c_k and c_{k+1}, and since $U_1(\cdot)$ is convex, $\bar{x}_1(c_k)/\bar{p}_1(c_k)\leqq\tilde{x}_1(c_k)/\tilde{p}_1(c_k)\leqq\bar{x}_1(c_{k+1})/\bar{p}_1(c_{k+1})$, as illustrated in Figure 5. But $\bar{p}_1(\cdot)$ is continuous and monotone, and hence $\bar{x}_1(\cdot)/\bar{p}_1(\cdot)$ is continuous and monotone, implying that

$$\max_k |\bar{x}_1(c_{k+1})/\bar{p}_1(c_{k+1})-\bar{x}_1(c_k)/\bar{p}_1(c_k)|\to 0$$

as $N\to\infty$, and hence establishing uniform convergence.

Part II: Construction of an approximating mechanism for $\delta<1$.

Consider the following system of $8N$ equations in the $8N$ unknowns $\{\hat{x}_1(a_k), \hat{x}_1(c_k), \hat{p}_1(a_k), \hat{p}_1(c_k), \hat{g}(a_k), \hat{g}(c_k), \hat{\beta}(c_k), \hat{p}_0(c_k)\}_{k=0}^{N-1}$:

$$\hat{x}_1(a_k)-\hat{x}_1(a_{k+1})-a_{k+1}[\hat{p}_1(a_k)-\hat{p}_1(a_{k+1})]=0 \tag{B.9a}$$

$$\hat{x}_1(c_k)-\hat{x}_1(c_{k+1})-c_{k+1}[\hat{p}_1(c_k)-\hat{p}_1(c_{k+1})]=0 \tag{B.9b}$$

$$-\hat{p}_1(a_k)+1-F_2(\hat{\beta}(a_k))+\delta[F_2(\hat{\beta}(a_k))-F_2(\hat{g}(a_k))]=0 \tag{B.9c}$$

$$-\hat{p}_1(c_k)+1-F_2(\hat{\beta}(c_k))+\delta[F_2(\hat{\beta}(c_k))-F_2(\hat{g}(c_k))]=0 \tag{B.9d}$$

$$-\hat{x}_1(a_k)+[1-F_2(\hat{\beta}(a_k))]\hat{p}_0(a_k)+\delta[F_2(\hat{\beta}(a_k))-F_2(\hat{g}(a_k))]\hat{g}(a_k)=0 \tag{B.9e}$$

$$-\hat{x}_1(c_k)+[1-F_2(\hat{\beta}(c_k))]\hat{p}_0(c_k)+\delta[F_2(\hat{\beta}(c_k))-F_2(\hat{g}(c_k))]\hat{g}(c_k)=0 \tag{B.9f}$$

$$-\tilde{B}_1(a_{k+1})+\tilde{B}_1(a_k)+[F_1(a_{k+1})-F_1(a_k)]\{\hat{x}_1(a_k)-\delta[1-F_2(\hat{g}(a_k))]\hat{g}(a_k)$$
$$-(1-\delta)[1-F_2(\hat{\beta}(a_k))]\hat{\beta}(a_k)\}=0 \tag{B.9g}$$

$$-\tilde{B}_1(c_{k+1})+\tilde{B}_1(c_k)+[F_1(c_{k+1})-F_1(c_k)]\{\hat{x}_1(c_k)-\delta[1-F_2(\hat{g}(c_k))]\hat{g}(c_k)$$
$$-(1-\delta)[1-F_2(\hat{\beta}(c_k))]\hat{\beta}(c_k)\}=0. \tag{B.9h}$$

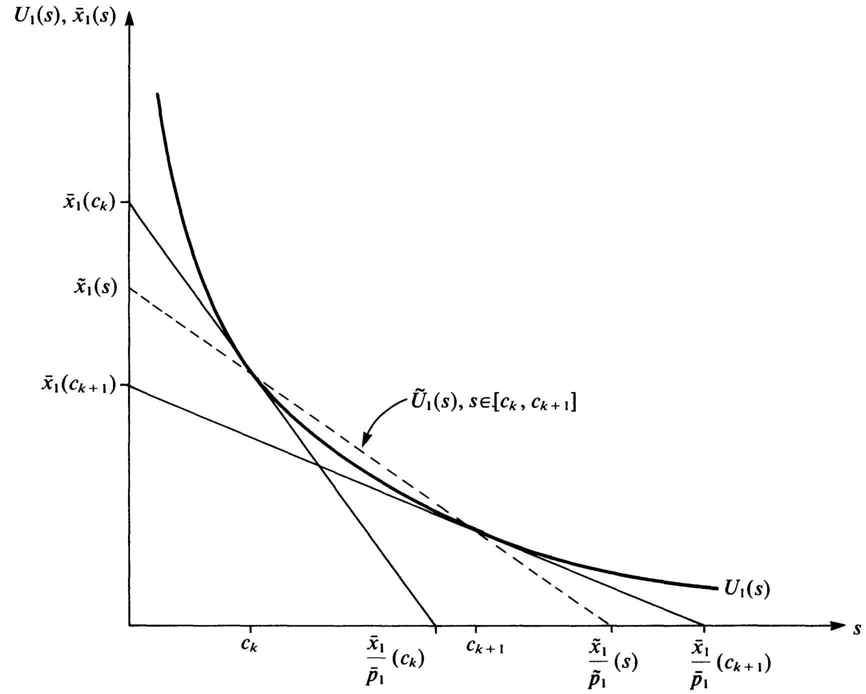


FIGURE 5

Graphical interpretation of $\bar{x}_1(s)$ and $\bar{x}_1(s)/\bar{p}_1(s)$.

for $k=0,\ldots,N-1$. As before, $\hat{\beta}(a_k)\equiv\hat{\beta}(c_k)$ and $\hat{p}_0(a_k)\equiv\hat{p}_0(c_k)$. $\tilde{B}_1(a_k)$ and $\tilde{B}_1(c_k)$ are constants determined in Part I. $\bar{p}_1(c_N)$ and $\bar{x}_1(c_N)$ are assigned the boundary values of zero.

A solution to (B.9) has the two-price interpretation if, in addition, $\hat{g}(c_k)\leqq\hat{\beta}(c_k)\leqq 1$ for $k=0,\ldots,N-1$. In the initial period, sellers with valuations in $[a_k, a_{k+1})\cup[c_k, c_{k+1})$ will be required to pool by offering the price $\hat{p}_0(c_k)$. In the following period (which is now discounted by δ), sellers in $[a_k, a_{k+1})$ will be required to offer $\hat{g}(a_k)$ and sellers in $[c_k, c_{k+1})$ will offer $\hat{g}(c_k)$. The probability-of-trade calculations in (B.9c–d) and the revenue calculations in (B.9e–f) will be justified provided that a buyer with valuation $\hat{\beta}(c_k)$ is indifferent between the initial and second offers, i.e.

$$\hat{\beta}(c_k)-\hat{p}_0(c_k)=\delta\{\hat{\beta}(c_k)-\tilde{\theta}(a_k)\hat{g}(a_k)-[1-\tilde{\theta}(c_k)]\hat{g}(c_k)\}. \tag{B.10}$$

However, substituting (B.9e) and (B.9f) into (B.9g) and (B.9h), respectively, and adding the resulting two equations implies (B.10).

It should be recalled that, in Part I, we constructed a solution to (B.9) for $\delta=1$ and arbitrary grids. The implicit function theorem will immediately imply the existence of solutions to (B.9) for all δ contained in a non-empty interval $(\tilde{\delta}_N, 1]$, provided that the Jacobian (with respect to the $8N$ unknowns) is non-zero. Voluminous calculations establish that this determinant, evaluated at $\delta=1$, equals a (non-zero) scalar multiple of

$$\prod_{k=0}^{N-1}[1-F_2(\tilde{\beta}(c_k))]f_2(\tilde{\beta}(c_k))[\tilde{g}(c_k)-\tilde{g}(a_k)]$$
$$\times[1-F_2(\tilde{g}(a_k))-f_2(\tilde{g}(a_k))(\tilde{g}(a_k)-a_{k+1})]$$
$$\times[1-F_2(\tilde{g}(c_k))-f_2(\tilde{g}(c_k))(\tilde{g}(c_k)-c_{k+1})].$$

Since $\tilde{\beta}(c_k)<1$ for all k, we have $1-F_2(\tilde{\beta}(c_k))>0$. Also $f_2(\tilde{\beta}(c_k))>0$. Observe that $1-F_2(\tilde{g}(c_k))-f_2(\tilde{g}(c_k))\times(\tilde{g}(c_k)-c_{k+1})=0$ if and only if $\tilde{g}(c_k)=g^*(c_{k+1})$ (and analogously for a_k). Judicious choice of the grid $\{c_k\}_{k=1}^{N-1}$ together with Lemma 4.2 (ii) assure that this is not the case for any c_k; meanwhile, Lemma 4.2 (ii) also assures that there exists $\hat{N}\geqq\tilde{N}$ such that this is not the case for any a_k, whenever $N\geqq\hat{N}$.

Finally, recall that for $N\geqq\tilde{N}$, $\tilde{\beta}(c_k)>\tilde{g}(c_k)+\varepsilon/2$ or $\tilde{x}_1(c_k)/\tilde{p}_1(c_k)>\tilde{p}_0(c_k)+\varepsilon/2$, for every $k=0,\ldots,N-1$. Since each of $\hat{\beta}(\cdot)$, $\hat{g}(\cdot)$, $\hat{x}_1(\cdot)/\hat{p}_1(\cdot)$ and $\hat{p}_0(\cdot)$ are continuous in δ, there exists for each N a value $\hat{\delta}_N\in(\tilde{\delta}_N, 1)$ such that $\hat{\beta}(c_k)<1$, and $\hat{\beta}(c_k)>\hat{g}(c_k)$ or $\hat{x}_1(c_k)/\hat{p}_1(c_k)>\hat{p}_0(c_k)$, for all $\delta\in(\hat{\delta}_N, 1]$ and all $k=0,\ldots,N-1$.

Part III: Construction of an approximating mechanism for $\delta<1$ and $\lambda>0$.

Consider the artificial game in which the seller must select from a menu of $2N$ price paths: $\hat{p}_0(c_k)$ in the initial period, followed by $\hat{g}(a_k)$ in all subsequent periods; and $\hat{p}_0(c_k)$ followed by $\hat{g}(c_k)$ forever ($k=0,\ldots,N-1$). If $\{\hat{p}_0(c_k), \hat{g}(a_k), \hat{g}(c_k)\}_{k=0}^{N-1}$ solved (B.9), it is incentive compatible for every seller in $[a_k, a_{k+1})$ and $[c_k, c_{k+1})$ to select her assigned price path. We will now modify the menu by replacing the price paths assigned to $[a_0, a_1)$ and $[c_0, c_1)$. Each seller in $[c_0, c_1)$ will offer p_0^* in the initial period, followed by a constant price path of g_c^* in all subsequent periods. Each seller in $[a_0, a_1)$ will also offer p_0^* in the initial period, but will then follow an exponentially-descending price path in subsequent periods. In particular, if $s\in[a_0, a_1)$, the seller will charge $e^{-\lambda(m-1)z}\,g_a^*$ in all periods $m\geqq 1$ that this price exceeds s, and will charge 1 in all subsequent periods. Let us define $W(g_a^*, \lambda, \beta^*; s)$ to be the net present value of utility to the seller of valuation s from charging an initial price of g_a^* and cutting price by a factor of $e^{-\lambda z}$ in each subsequent period (until price drops below s), if the initial buyer distribution is $F_2(\cdot)$ truncated at β^* and if the buyer purchases optimally (under the beliefs that the seller distribution is $F_1(\cdot)$ truncated at a_1).

In order to preserve the incentive compatibility of the modified menu, it is sufficient to guarantee that the following system of four equations:

$$\beta^*-p_0^*-\delta\{\beta^*-\tilde{\theta}(a_0)g_a^*-[1-\tilde{\theta}(a_0)]g_c^*\}=0 \tag{B.11a}$$

$$[1-F_2(\beta^*)][p_0^*-a_N]+\delta[F_2(\beta^*)-F_2(g_c^*)][g_c^*-a_N]-[\hat{x}_1(a_{N-1})-a_N\hat{p}_1(a_{N-1})]=0 \tag{B.11b}$$

$$[1-F_2(\beta^*)][p_0^*-c_1]+\delta[F_2(\beta^*)-F_2(g_c^*)][g_c^*-c_1]-[\hat{x}_1(c_1)-c_1\hat{p}_1(c_1)]=0 \tag{B.11c}$$

$$[1-F_2(\beta^*)][p_0^*-a_1]+\delta W(g_a^*, \lambda, \beta^*, a_1)-[\hat{x}_1(a_1)-a_1\hat{p}_1(a_1)]=0, \tag{B.11d}$$

is solved and that the implied $\bar{p}_1(\cdot)$ function on $[a_0, a_1)$ satisfies $\lim_{s\uparrow a_1}\bar{p}_1(s)\geqq\hat{p}_1(a_1)$. The latter inequality will hold for a rectangle of pairs (λ, δ), since it holds strictly when $\lambda=0$ and $\delta=1$ (i.e. $\tilde{p}_1(a_0)>\tilde{p}_1(a_1)$). Equation (B.11a) requires the buyer to respond optimally to p_0^*, (B.11b) makes $a_N(=c_0)$ indifferent between the price paths for $[a_{N-1}, a_N)$ and $[c_0, c_1)$, (B.11c) makes c_1 indifferent between the paths for $[c_0, c_1)$ and $[c_1, c_2)$, and (B.11d) makes a_1 indifferent between the paths for $[a_0, a_1)$ and $[a_1, a_2)$.

It should be observed that, in Part I, we constructed a solution to (B.11) for $(\delta, \lambda) = (1, 0)$ and for arbitrary grids. (Indeed, the solution was $(\beta^*, p_0^*, g_a^*, g_c^*) = (\tilde{\beta}(c_0), \tilde{p}_0(c_0), \tilde{g}(a_0), \tilde{g}(c_0))$.) Also, we established in Part II that for each N there exists $\hat{\delta}_N < 1$ such that, for every $\delta \in (\hat{\delta}_N, 1]$, the six terms $\hat{p}_1(c_1)$, $\hat{x}_1(c_1)$, $\hat{p}_1(a_{N-1})$, $\hat{x}_1(a_{N-1})$, $\hat{p}_1(a_1)$ and $\hat{x}_1(a_1)$ which appear in (B.11) may all be parameterized with respect to δ. Thus, the implicit function theorem will immediately imply the existence of solutions to (B.11) for all pairs (δ, λ) contained in a non-empty rectangle $(\delta_N^*, 1] \times [0, \lambda_N^*)$, where $\delta_N^* \geqq \hat{\delta}_N$, provided that the system (B.11) is continuously differentiable in the four unknowns $(\beta^*, p_0^*, g_a^*, g_c^*)$ and that the Jacobian is non-zero.

The continuous differentiability of the system (B.11) follows from the fact that $W(g_a^*, \lambda, \beta^*; s)$ is continuously differentiable in g_a^* (which is shown, somewhat laboriously, in the working paper precursor (Ausubel and Deneckere (1988*b*)). Direct calculation of the Jacobian at $(\delta, \lambda) = (1, 0)$ yields the value $-f_2(\tilde{g}(c_0))\tilde{\theta}(c_0)[\tilde{g}(c_0) - \tilde{g}(a_0)][1 - F_2(\tilde{g}(a_0)) - f_2(\tilde{g}(a_0))(\tilde{g}(a_0) - a_1)]$. As argued in Part II, Lemma 4.2 (ii) assures that this Jacobian is non-zero for $N \geqq \hat{N}$. Consequently, for each $N \geqq \hat{N}$, there exists $\delta_N^* < 1$ and $\lambda_N^* > 0$ such that the implicit function theorem is applicable on the rectangle $(\delta_N^*, 1] \times [0, \lambda_N^*)$.

Finally, recall that for $N \geqq \tilde{N}$, $\tilde{\beta}(c_0) > \tilde{g}(c_0) + \varepsilon/2$ or $\tilde{x}_1(c_0)/\tilde{p}_1(c_0) > \tilde{p}_0(c_0) + \varepsilon/2$ and also $\tilde{\beta}(c_0) < 1$. Since each of β^*, p_0^*, g_a^* and g_c^* are jointly continuous in δ and λ, there exist for every $N \geqq \hat{N}$ values $\delta_N^{**} \in (\delta_N^*, 1)$ and $\lambda_N^{**} \in (0, \lambda_N^*)$ such that $g_c^* < \beta^* < 1$ for all pairs $(\delta, \lambda) \in (\delta_N^{**}, 1] \times [0, \lambda_N^{**})$. Thus, for every (δ, λ) contained in this rectangle, we have shown the existence of an approximating mechanism, $p_{N,\delta,\lambda}^*$, which has the two-price interpretation.

Part IV: Construction of the stationary sequential equilibrium.

Let us begin the construction of equilibria by assuring that our initial choices for the grid $\{c_k\}_{k=0}^N$ had the property that the N implied values $\tilde{p}_0(c_0), \ldots, \tilde{p}_0(c_{N-1})$ were all different. (Observe that for generic choices of the grid, the N values are different, but in the non-generic case where $\tilde{p}_0(c_j) = \tilde{p}_0(c_k)$ for some $j \neq k$, it will be necessary to perturb the grid.) Also, if necessary, redefine δ_N^{**} closer to 1 and λ_N^{**} closer to 0, so that for all pairs $(\delta, \lambda) \in (\delta_N^{**}, 1] \times [0, \lambda_N^{**})$, the N values p_0^*, $\hat{p}_0(c_2), \ldots, \hat{p}_0(c_N)$ are all different. This will enable the seller's initial offer, in the equilibrium we construct below, to fully convey the fact that

$$s \in [a_k, a_{k+1}) \cup [c_k, c_{k+1}).$$

For any $\delta < 1$, let σ_δ be a weak-Markov equilibrium in the seller-offer game with discount factor δ between periods, where the seller's valuation is commonly known to equal zero and the buyer's valuation is distributed according to $F_2(\cdot)$. Let σ_δ^b denote the buyer's strategy in σ_δ. We may now specify the equilibrium strategies:

Seller's strategy:

- If there has been no prior seller deviation, follow the price path specified for seller type s in the artificial game of the first paragraph of Part III. If $s \in [\hat{s}, 1]$, charge a price of 1 forever.
- If there has been an undetectable seller deviation (but no detectable seller deviation), follow the price path which maximizes utility subject to keeping the deviation undetectable. If that involves pricing below cost, charge a price of 1 instead.
- If there has been a detectable seller deviation, optimize against a buyer strategy of σ_δ^b.

Buyer's strategy:

- If there has been no prior detectable seller deviation, optimize for buyer type b against the seller's chosen price path and the induced beliefs about the seller's type.
- If there has been a detectable seller deviation, update beliefs to $s = 0$ and maintain these beliefs forever after. Accept or reject using the strategy σ_δ^b.

For sufficiently large N, we constructed in Part III a rectangle $(\delta_N^{**}, 1] \times [0, \lambda_N^{**})$ where the implicit function theorem was applicable. Select any $\lambda_N \in (0, \lambda_N^{**})$. Paralleling the argument of Facts 2 and 3 in Section 4 of Ausubel and Deneckere (1992), there exists $\delta_N^{***}(\lambda_N) \in (\delta_N^{**}, 1)$ such that, if δ_N satisfies $\delta_N^{***}(\lambda_N) < \delta_N < 1$, the weak-Markov strategy σ_δ^b is sufficiently severe to deter all detectable seller deviations. It is straightforward to verify that our mechanism construction precludes undetectable deviations as well.

Selecting λ_N and δ_N so that $\lambda_N \downarrow 0$ and $\delta_N \uparrow 1$, we see that the sequence of probability-of-trade functions $\{p_{N,\delta_N,\lambda_N}^*\}_{N=1}^\infty$ induced by our constructed sequential equilibria converges in measure to our original p, as $N \to \infty$. Thus, the ex ante efficient mechanism which places weight $\lambda \in (\lambda_s, 1)$ on the seller is implementable by sequential equilibria in the seller-offer game.

To implement the mechanism with weight λ_s (or 1), consider any sequence $\{\lambda^j\}_{j=1}^{\infty} \subset (\lambda_s, 1)$ converging to λ_s (or 1). Each mechanism with weight λ^j is implementable by the above argument; a diagonal argument then shows implementability for λ_s (or 1). The buyer result is proven analogously. ||

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REFERENCES

AUSUBEL, L. and DENECKERE, R. (1988*a*), "Stationary Sequential Equilibria in Bargaining with Two-Sided Incomplete Information" (CMSEMS Discussion Paper No. 784, Northwestern University).

AUSUBEL, L. and DENECKERE, R. (1988*b*), "Efficient Sequential Bargaining" (CMSEMS Discussion Paper No. 804, Northwestern University).

AUSUBEL, L. and DENECKERE, R. (1989*a*), "Reputation in Bargaining and Durable Goods Monopoly", *Econometrica*, **57**, 511-531.

AUSUBEL, L. and DENECKERE, R. (1989*b*), "A Direct Mechanism Characterization of Sequential Bargaining with One-Sided Incomplete Information", *Journal of Economic Theory*, **48**, 18-46.

AUSUBEL, L. and DENECKERE, R. (1992), "Durable Goods Monopoly with Incomplete Information", *Review of Economic Studies*, **59**, 795-812.

CHATTERJEE, K. and SAMUELSON, W. (1983), "Bargaining Under Incomplete Information", *Operations Research*, **31**, 835-851.

CRAMTON, P. (1984), "Bargaining with Incomplete Information: An Infinite-Horizon Model with Continuous Uncertainty", *Review of Economic Studies*, **51**, 579-593.

CRAMTON, P. (1985), "Sequential Bargaining Mechanisms", in A. Roth (ed.), *Game Theoretic Models of Bargaining* (Cambridge: Cambridge University Press), 149-179.

FUDENBERG, D., LEVINE, D. and TIROLE, J. (1985), "Infinite Horizon Models of Bargaining with One-Sided Incomplete Information", in A. Roth (ed.), *Game Theoretic Models of Bargaining* (Cambridge: Cambridge University Press), 73-98.

GUL, F. and SONNENSCHEIN, H. (1988), "On Delay in Bargaining with One-Sided Uncertainty", *Econometrica*, **56**, 601-612.

GUL, F., SONNENSCHEIN, H. and WILSON, R. (1986), "Foundations of Dynamic Monopoly and the Coase Conjecture", *Journal of Economic Theory*, **39**, 155-190.

MYERSON, R. (1985), "Analysis of Two Bargaining Problems with Incomplete Information", in A. Roth (ed.), *Game Theoretic Models of Bargaining* (Cambridge: Cambridge University Press), 115-147.

MYERSON, R. and SATTERTHWAITE, M. (1983), 'Efficient Mechanisms for Bilateral Trading", *Journal of Economic Theory*, **28**, 265-281.

RUBINSTEIN, A. (1982), "Perfect Equilibrium in a Bargaining Model", *Econometrica*, **50**, 97-109.

RUBINSTEIN, A. (1987), "A Sequential Strategic Theory of Bargaining", in T. Bewley (ed.), *Advances in Economic Theory* (Cambridge: Cambridge University Press), 197-224.

WILLIAMS, S. (1987), "Efficient Performance in Two Agent Bargaining", *Journal of Economic Theory*, **41**, 154-172.

WILSON, R. (1987), "Game-Theoretic Analyses of Trading Processes", in T. Bewley (ed.), *Advances in Economic Theory* (Cambridge: Cambridge University Press), 33-70.