

# Outline for Static Games of Complete Information

- I. Definition of a game
- II. Examples
- III. Definition of Nash equilibrium
- IV. Examples, continued
- V. Iterated elimination of dominated strategies
- VI. Mixed strategies
- VII. Existence theorem on Nash equilibria
- VIII. The Hotelling model and extensions

Copyright © 2009 by Lawrence M. Ausubel

**Definition:** An  $n$ -player, **static game** of complete information consists of an  $n$ -tuple of strategy sets and an  $n$ -tuple of payoff functions, denoted by  $G = \{S_1, \dots, S_n; u_1, \dots, u_n\}$

$S_i$ , the **strategy set** of player  $i$ , is the set of all permissible moves for player  $i$ . We write  $s_i \in S_i$  for one of player  $i$ 's strategies.

$u_i$ , the **payoff function** of player  $i$ , is the utility, profit, etc. for player  $i$ , and depends on the strategies chosen by all the players:  $u_i(s_1, \dots, s_n)$ .

## Example: Prisoners' Dilemma

		Prisoner II	
		Remain Silent	Confess
Prisoner I	Remain Silent	-1 , -1	-5 , 0
	Confess	0 , -5	-4 , -4

## Example: Battle of the Sexes

		F	
		Boxing	Ballet
M	Boxing	2 , 1	0 , 0
	Ballet	0 , 0	1 , 2

**Definition:** A Nash equilibrium of  $G$  (in pure strategies) consists of a strategy for every player with the property that no player can improve her payoff by unilaterally deviating:

$$(s_1^*, \dots, s_n^*) \text{ with the property that, for every player } i:$$

$$u_i(s_1^*, \dots, s_{i-1}^*, s_i^*, s_{i+1}^*, \dots, s_n^*)$$

$$\geq u_i(s_1^*, \dots, s_{i-1}^*, s_i, s_{i+1}^*, \dots, s_n^*)$$

for all  $s_i \in S_i$ .

Equivalently, a Nash equilibrium is a mutual best response. That is, for every player  $i$ ,  $s_i^*$  is a solution to:

$$s_i^* \in \arg \max_{s_i \in S_i} \{u_i(s_1^*, \dots, s_{i-1}^*, s_i, s_{i+1}^*, \dots, s_n^*)\}$$

## Example: Prisoners' Dilemma

		<b>Prisoner II</b>	
		Remain Silent	Confess
<b>Prisoner I</b>	Remain Silent	-1 , -1	-5 , 0
	Confess	0 , -5	-4 , -4

# Example: Battle of the Sexes

		<b>F</b>	
		Boxing	Ballet
<b>M</b>	Boxing	2 , 1	0 , 0
	Ballet	0 , 0	1 , 2

## Cournot (1838) Model of Oligopoly

- (a)  $n$  firms
- (b) Each firm  $i$  has a constant marginal (and average) cost of  $c_i$
- (c) Inverse aggregate demand function of  $P(Q)$
- (d) Each firm simultaneously and independently selects a strategy consisting of a **quantity**  $q_i \in [0, a]$  (where  $P(a) = 0$ )

Then, with two firms, the payoff functions are:

$$\pi_1(q_1, q_2) = q_1 P(q_1 + q_2) - c_1 q_1$$

$$\pi_2(q_1, q_2) = q_2 P(q_1 + q_2) - c_2 q_2 .$$

and the strategy sets are:

$$S_1 = [0, a] \qquad S_2 = [0, a]$$

It is often also convenient to assume a common marginal cost (i.e.,  $c_1 = c = c_2$ ) and a linear demand curve  $P(Q) = a - Q$ .

# Solution of Cournot Model with Two Firms

$(q_1^*, q_2^*)$  is a Nash equilibrium if and only if:

$$q_1^* \text{ solves } \max_{q_1} \{q_1 [P(q_1 + q_2^*) - c]\}$$

and

$$q_2^* \text{ solves } \max_{q_2} \{q_2 [P(q_1^* + q_2) - c]\}.$$

With  $P(Q) = a - Q$ , we get first order conditions:

$$q_1(-1) + a - q_1 - q_2^* - c|_{q_1=q_1^*} = 0$$

$$(1) \quad a - 2q_1^* - q_2^* = c$$

$$\text{and: } q_2(-1) + a - q_1^* - q_2 - c|_{q_2=q_2^*} = 0$$

$$(2) \quad a - q_1^* - 2q_2^* = c$$

Subtracting (1) - (2) gives:

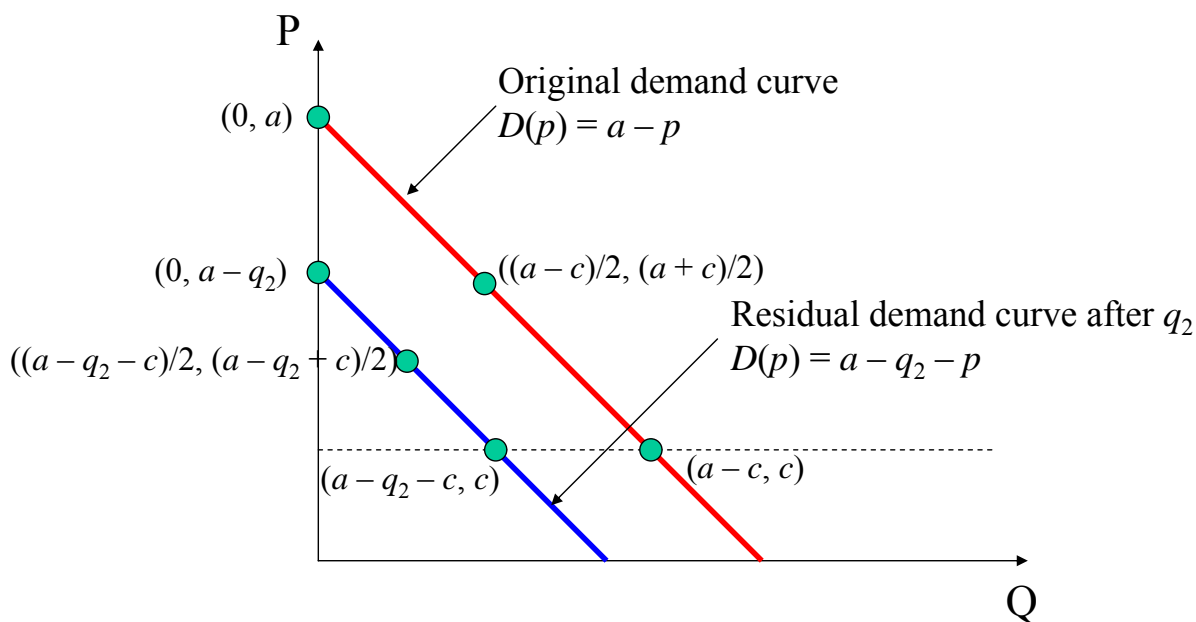
$$q_2^* - q_1^* = 0$$

Substituting  $q_2^* = q_1^*$  into (1) gives:

$$a - 2q_1^* - q_1^* = c$$

$$q_1^* = (a - c) / 3 ; \quad q_2^* = (a - c) / 3 .$$

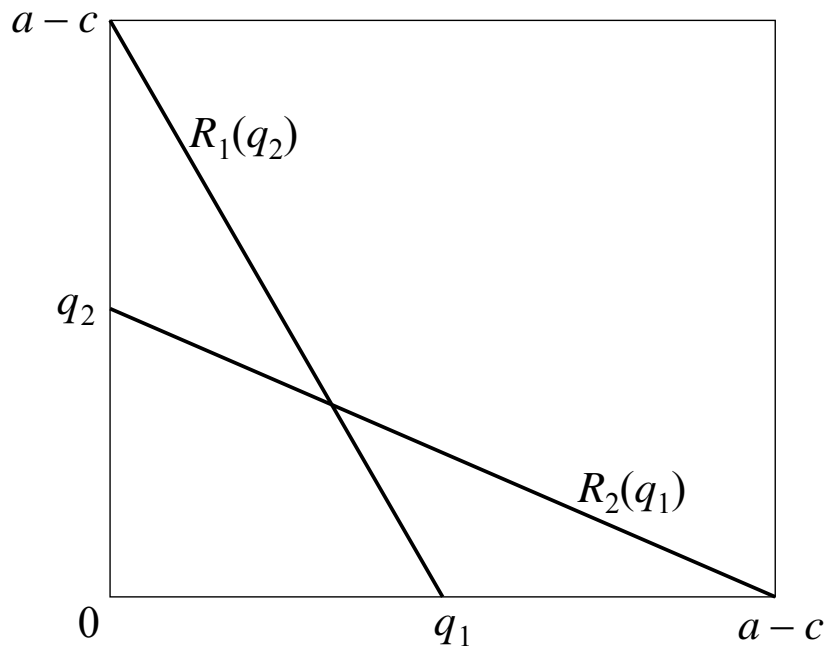
## Best Response for Firm 1 to $q_2$



$$R_1(q_2) = (a - q_2 - c)/2$$

Similarly, the best response for firm 2 to  $q_1$  is:

$$R_2(q_1) = (a - q_1 - c)/2$$



Cournot Duopoly: Best Response Functions

## Bertrand (1883) Model of Oligopoly

- (a)  $n$  firms
- (b) Each firm  $i$  has a constant marginal (and average) cost of  $c_i$
- (c) Aggregate demand function of  $Q(P)$
- (d) Each firm simultaneously and independently selects a strategy consisting of a **price**  $p_i \in [0, a]$  (where  $Q(a) = 0$ )

Then, with two firms, the payoff functions are:

$$\pi_1(p_1, p_2) = \begin{cases} Q(p_1)[p_1 - c_1], & \text{if } p_1 < p_2 \\ \frac{1}{2}Q(p_1)[p_1 - c_1], & \text{if } p_1 = p_2 \\ 0, & \text{if } p_1 > p_2 \end{cases}$$

and

$$\pi_2(p_1, p_2) = \begin{cases} Q(p_2)[p_2 - c_2], & \text{if } p_2 < p_1 \\ \frac{1}{2}Q(p_2)[p_2 - c_2], & \text{if } p_2 = p_1 \\ 0, & \text{if } p_2 > p_1 \end{cases}$$

# Bertrand (1883) Model of Oligopoly

As in the Cournot game, the strategy sets are:

$$S_1 = [0, a] \quad S_2 = [0, a]$$

and it is again usually convenient to assume a common marginal cost (i.e.,  $c_1 = c = c_2$ ).

## Solution of Bertrand game with two firms and common marginal cost $c_1 = c = c_2$ :

Observation 1: In any Nash equilibrium  $(p_1^*, p_2^*)$ , it must be the case that  $p_1^* \geq c$  and  $p_2^* \geq c$ .

Proof: Suppose otherwise. Without loss of generality, say  $p_1^* \leq p_2^*$  and  $p_1^* < c$ . Then firm 1 is currently earning strictly negative profits and could profitably deviate to  $p_1^* \geq c$  (thereby instead earning nonnegative profits).

# Bertrand (1883) Model of Oligopoly

Observation 2: In any Nash equilibrium  $(p_1^*, p_2^*)$ , it must be the case that  $p_1^* = p_2^*$ .

Proof: Suppose otherwise. Without loss of generality, say  $p_1^* < p_2^*$  (and  $p_1^* \geq c$ ). Then firm 2 is currently earning zero profits and, if  $p_1^* > c$ , firm 2 can profitably deviate to  $p_2^* = p_1^* - \varepsilon$ . Meanwhile, if  $p_1^* = c$ , firm 1 can profitably deviate to  $p_1^* = p_2^* - \varepsilon$ .

Observation 3: The unique Nash equilibrium is  $(p_1^*, p_2^*) = (c, c)$ .

Proof: By Observations 1 and 2, the only remaining possibility is  $p_1^* = p_2^* = p^* > c$ . Then each firm is currently earning profits of:

$$\frac{1}{2}Q(p^*)[p^* - c]$$

and either firm could profitably deviate to  $p^* - \varepsilon$  and thereby come arbitrarily close to earning:

$$Q(p^*)[p^* - c].$$

*Q.E.D.*

# The Pollution Game

Consumers have a choice of three different models of cars, which are identical in all respects except for price and emissions:

Model A:  $p_A = \$15,000$ ;  $e_A = 100$  units

Model B:  $p_B = \$16,000$ ;  $e_B = 10$  units

Model C:  $p_C = \$17,000$ ;  $e_C = 0$  units

A consumer's utility from using a car is given by:

$$U = v - p - E$$

where  $v$  = reservation value of a car;

$p$  = price paid for model bought;

$E = \sum_{i=1}^N e_i$  = aggregate emissions (over all consumers)  
where  $e_i = 100$  or  $10$  or  $0$ , depending on  
which model is purchased by consumer  $i$ .

## Dominated strategies:

Strategy  $s_i$  (strictly) **dominates** strategy  $s_i'$  if, for *all* possible strategy combinations of opponents,  $s_i$  yields a (strictly) higher payoff than  $s_i'$  to player  $i$ .

## Iterated elimination of strictly dominated strategies:

Eliminate all strategies which are dominated, relative to opponents' strategies which have not yet been eliminated.

## Player II

		Left	Right
Player I	Top	<del>1, 2</del>	<del>4, 1</del>
	Middle	3, 2	2, 1
	Bottom	<del>2, 1</del>	<del>1, 3</del>

Bottom is dominated by Middle (for Player I)

Right is dominated by Left (for Player II)

Top is dominated by Middle (for Player I)

## Results on Iterated Elimination of Strictly Dominated Strategies

**Proposition 1:** If iterated elimination of strictly dominated strategies yields a *unique* strategy  $n$ -tuple, then this strategy  $n$ -tuple is the *unique* Nash equilibrium (and it is a *strict* Nash equilibrium).

(Definition: A *strict* Nash equilibrium is a strategy  $n$ -tuple with the property that every unilateral deviation makes the deviator *strictly* worse off.)

**Proposition 2:** Every Nash equilibrium survives iterated elimination of strictly dominated strategies.

**Proposition 3:** Iterated elimination of strictly dominated strategies is order-independent.

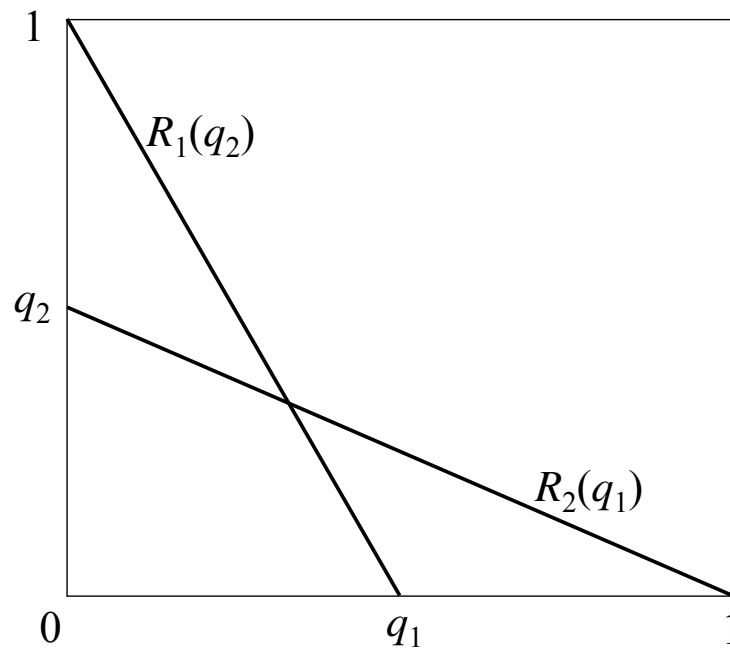
# Guess 2/3 of Average

## The Problem

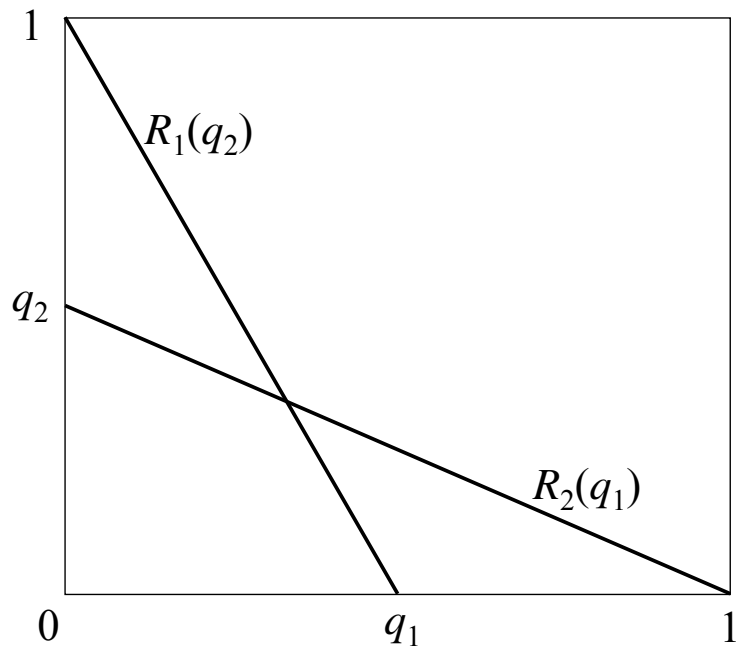
Each of you have to choose an integer between 0 and 100 in order to guess "2/3 of the average of the responses given by all students in the course".

Each student who guesses the integer which is 2/3 of the average of all the responses rounded up to the nearest integer, wins.

**What is your guess?**



**Cournot Duopoly: Best Response Functions**



- $q_1 > 1/2$  is strictly dominated by  $q_1 = 1/2$
- $q_2 > 1/2$  is strictly dominated by  $q_2 = 1/2$
- $q_1 < 1/4$  is strictly dominated by  $q_1 = 1/4$
- $q_2 < 1/4$  is strictly dominated by  $q_2 = 1/4$

## Example: Matching Pennies

<b>II</b>			
Heads      Tails			
<b>I</b>	Heads	1 , -1	-1 , 1
	Tails	-1 , 1	1 , -1

**Definition:** Let player  $i$  have  $K$  pure strategies available. Then a **mixed strategy** for player  $i$  is a probability distribution over those  $K$  strategies.

**Notation:**

Strategy set:

$$S_i = \{s_{i1}, \dots, s_{iK}\}$$

Mixed strategy:

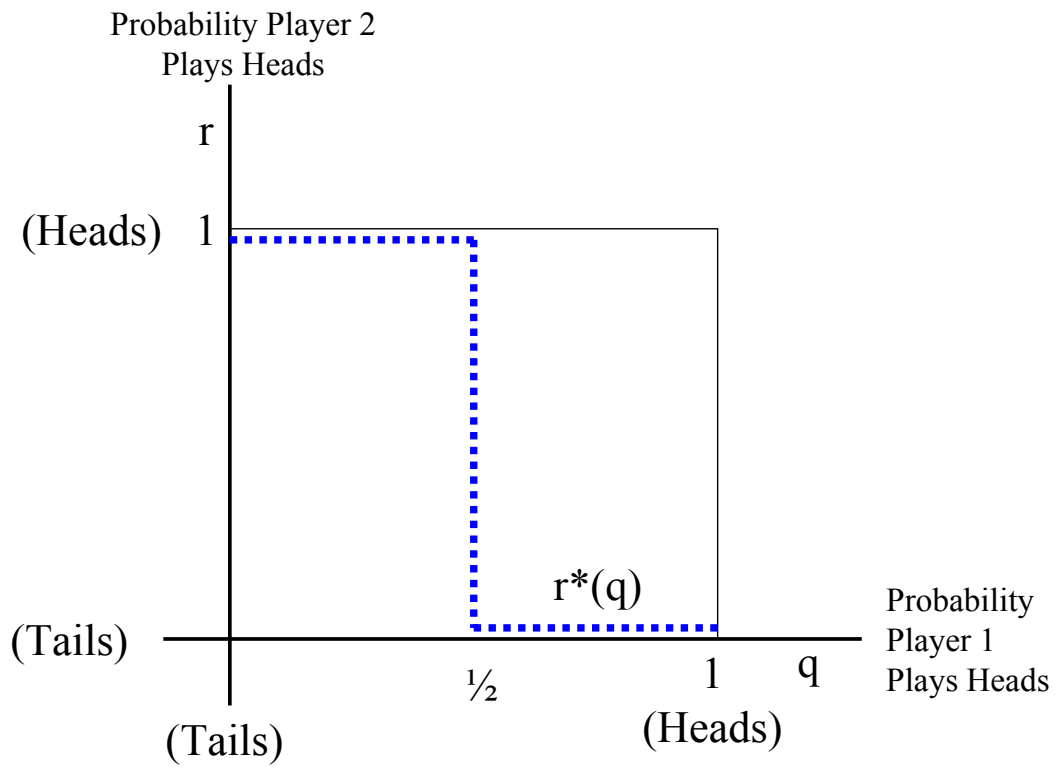
$$p_i = (p_{i1}, \dots, p_{iK})$$

such that  $\sum_{k=1}^K p_{ik} = 1$

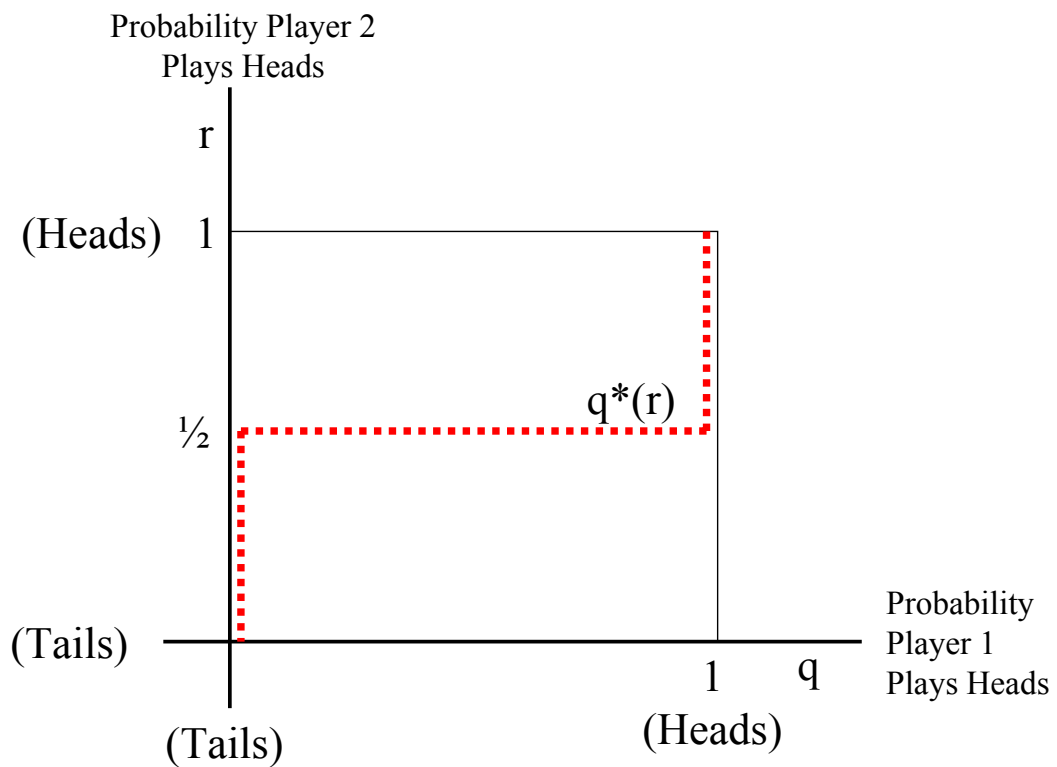
and each  $p_{ik}$  is between zero and one ( $0 \leq p_{ik} \leq 1$ ).

**Facts:**

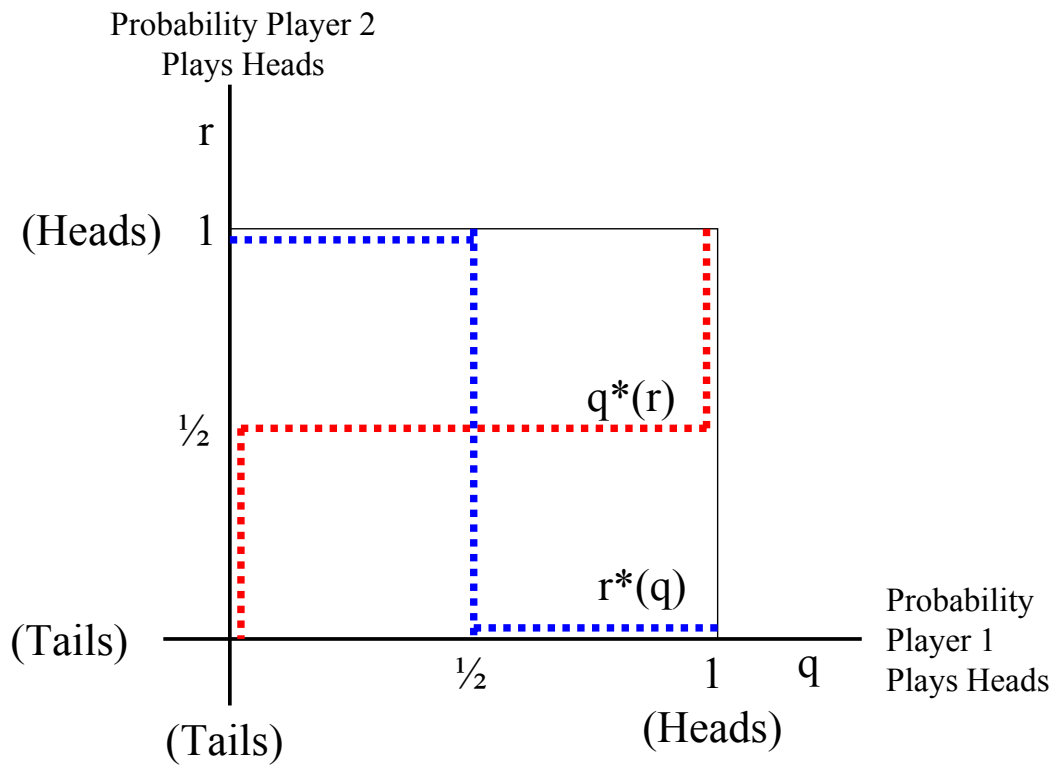
1. **Theorem** (Nash, 1950):  
Every finite game has at least one Nash equilibrium (when mixed strategies are permitted).
2. If, in a mixed-strategy Nash equilibrium, player  $i$  places positive probability on each of two strategies, then player  $i$  must be indifferent between these two strategies (i.e., they yield player  $i$  the same expected payoff).



Best response correspondence of Player 2



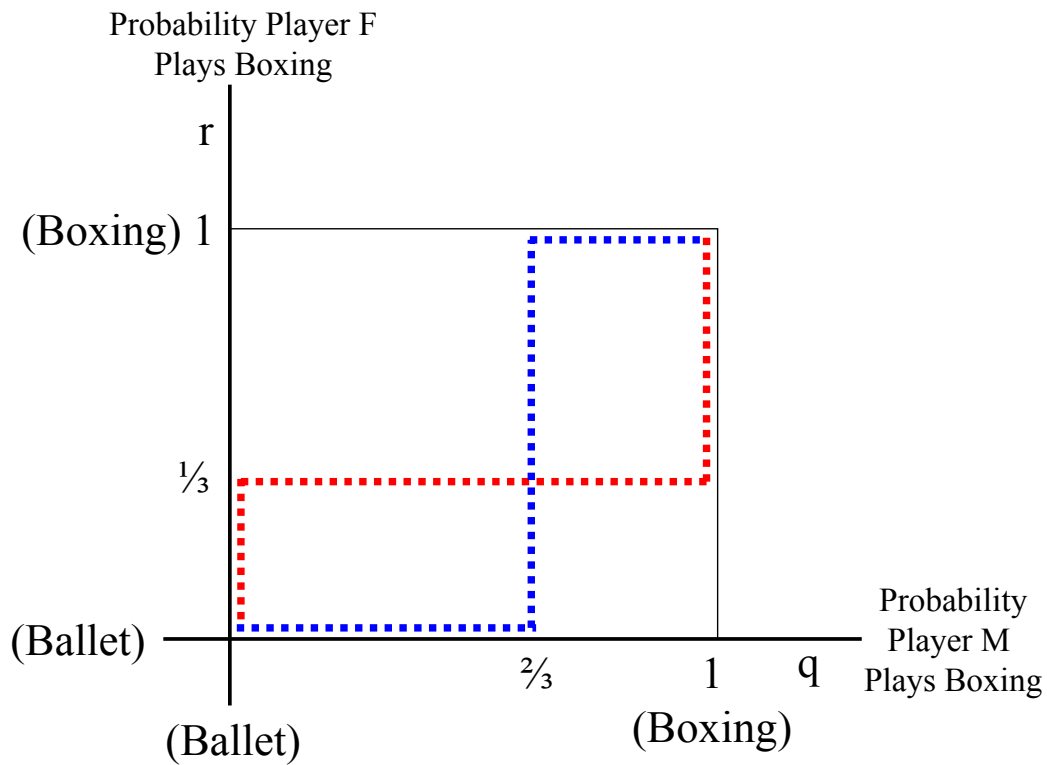
Best response correspondence of Player 1



Matching Pennies

## Example: Battle of the Sexes

		<b>F</b>	
		Boxing	Ballet
<b>M</b>	Boxing	2 , 1	0 , 0
	Ballet	0 , 0	1 , 2



Battle of the Sexes

**Correlated Equilibrium I:** Public Randomizing Device

Suppose that there exists a public randomizing device that comes up “heads”  $\frac{1}{2}$  the time and “tails”  $\frac{1}{2}$  the time. Then the players could agree to play {Boxing, Boxing} when “heads” and {Ballet, Ballet} when “tails”.

Example: Play {Boxing, Boxing} when the closing DJIA is an even number and play {Ballet, Ballet} when it is an odd number; achieving E payoffs of  $(\frac{3}{2}, \frac{3}{2})$  — better than the mixed strategy Nash equilibrium.

		<b>F</b>	
		Boxing	Ballet
<b>M</b>	Boxing	2 , 1	0 , 0
	Ballet	0 , 0	1 , 2

## Correlated Equilibrium II: Mediated communication

Consider the following game:

		<b>II</b>	
		L	R
<b>I</b>	T	5 , 1	0 , 0
	B	4 , 4	1 , 5

A public randomizing device enables us to obtain expected payoffs of (3, 3).

However, a mediator could randomize among three announcements — (T,L), (B,L) and (B,R). The mediator tells player I whether to play T or B (but *not* what he has told player II). Similarly, he tells player II whether to play L or R (and *not* what he has told player I). It can be shown: (i) No player has any incentive to deviate from these instructions; and (ii) E payoffs are now (10/3, 10/3)!

## Results on correlated equilibrium:

1. With attention limited to public randomizing devices, the set of outcomes of correlated equilibria is the convex combination of all (pure- and mixed-strategy) Nash equilibrium outcomes.
2. With mediated communication possible, one can sometimes construct correlated equilibria that outperform any convex combination of Nash equilibria: see for example the previous slide.

### **Brouwer Fixed Point Theorem:**

Suppose that  $X$  is a nonempty, compact, convex set in  $\mathbb{R}^n$ . Also suppose that the *function*  $f : X \rightarrow X$  is continuous. Then there exists a *fixed point* of  $f$ , i.e., a point  $x \in X$  such that  $x = f(x)$ .

### **Kakutani Fixed Point Theorem:**

Suppose  $X$  as above. Also suppose that the *correspondence*  $F : X \rightarrow X$  is nonempty and convex-valued, and that  $F(\cdot)$  has a closed graph. Then there exists a *fixed point* of  $F$ , i.e., a point  $x \in X$  such that  $x \in F(x)$ .

### **Notes:**

(1) The correspondence  $F(\cdot)$  is said to have a *closed graph* if, simply, the graph of  $F(\cdot)$  is a closed set. That is,  $F(\cdot)$  has a closed graph if it has the property that whenever the sequence  $(x^n, y^n) \rightarrow (x, y)$ , with  $y^n \in F(x^n)$  for every  $n$ , then  $y \in F(x)$ .

Essentially the same as upper hemicontinuity (u.h.c.).

(2) The best-response correspondence  $BR_i(\cdot)$  of each player  $i$  has a closed graph, by the following argument.

Suppose that there is a sequence  $(x^n, y^n) \rightarrow (x, y)$  such that  $y^n \in BR_i(x^n)$  for every  $n$ , but  $y \notin BR_i(x)$ . Then there exists  $\varepsilon > 0$  and  $y' \neq y$  such that:

$$u_i(y', x) > u_i(y, x) + \varepsilon.$$

But this contradicts:

$$u_i(y', x^n) \leq u_i(y^n, x^n), \text{ for every } n.$$

# For best response to exist, need maximum to exist

- Continuous function on compact set has a maximum; hence, require:

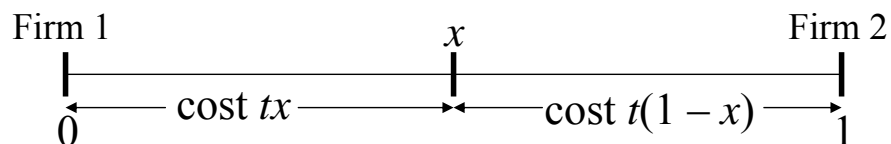
– closed  or no max

– bounded  or no max

– continuous  or no max

## Product Differentiation: The Hotelling Model

Consumers are uniformly distributed on the interval  $[0, 1]$ . There are two firms, located at  $x = 0$  and  $x = 1$ , which each produce the same physical good at marginal cost of  $c$ . Consumers have transportation cost  $t$  per unit of distance.



Each consumer consumes 0 or 1 units of the good:

$$u(0) = 0; \quad u(1) = v.$$

If firm 1 charges  $p_1$  and firm 2 charges  $p_2$ , the consumer located at  $x$  gets  $v - p_1 - tx$  from purchasing at firm 1 and gets  $v - p_2 - t(1 - x)$  from purchasing at firm 2.

Let  $\tilde{x}$  denote the customer who is indifferent between purchasing at firm 1 and firm 2. Then:

$$v - p_1 - t\tilde{x} = v - p_2 - t(1 - \tilde{x})$$

$$2t\tilde{x} = t + p_2 - p_1$$

$$\tilde{x} = \frac{1}{2} + \frac{p_2 - p_1}{2t}.$$

The profits of firm 1 are given by:

$$\pi_1(p_1, p_2) = [p_1 - c] \tilde{x} = [p_1 - c] \left[ \frac{1}{2} + \frac{p_2 - p_1}{2t} \right].$$

The profits of firm 2 are given by:

$$\pi_2(p_1, p_2) = [p_2 - c][1 - \tilde{x}] = [p_2 - c] \left[ \frac{1}{2} - \frac{p_2 - p_1}{2t} \right].$$

These imply the first-order conditions of:

$$(1) \quad t + c + p_2^* - 2p_1^* = 0$$

$$(2) \quad t + c + p_1^* - 2p_2^* = 0.$$

Solving yields:

$$p_1^* = t + c; \quad p_2^* = t + c.$$