

Problem Set 8

1. “The Game of Whether or Not to Write Your Name on the Exam Booklet.”

Consider a game with the following story behind it. Each of N students in a final exam is faced with the choice of whether or not to write his name on his exam booklet. All other things being equal, a student prefers not writing his name as compared to writing his name, due to the demanding cost in time and energy of writing his name. If a given student is the only one to choose not to write his name, then it is obvious to the professor which exam booklet is associated with that student, and so his choice does not harm his grade. However, if two or more students choose not to write their names, then it is unclear which exam booklet is associated with which student, and so all of the students who did not write their names receive grades of “F” on the final.

To be more precise, the N students each simultaneously and independently select actions from the set {Write Name, Do Not Write Name}. For each student i , the payoff equals:

- 0, if student i wrote his name on the exam booklet, regardless of the actions of the other students;
- 1, if student i did not write his name on the exam booklet, and if every other student j ($j \neq i$) **did** write his name on the exam booklet;
- 10, if student i did not write his name on the exam booklet, and if some other student j ($j \neq i$) also **did not** write his name on the exam booklet.

Solve for **all** of the Nash equilibria of this game, demonstrating that each of these satisfies the definition of Nash equilibrium.

2. Consider the following game: Player 1 and player 2 simultaneously write down any positive integer. The player who writes down the higher number wins \$100; the player who writes down the smaller number wins nothing. In the event they write down the same number, they split the \$100.
- (a) Write down the strategy space, S_i , for each player, and the payoff function, $u_i(\cdot, \cdot)$, for each player.
 - (b) Prove that this game does not have any Nash equilibrium, and explain why the existence theorem for Nash equilibria does not apply to this game.
3. Consider the “Guess 2/3 of the Average” game discussed in class, but with the rules modified to say that “The student who guesses the integer which is closest to 2/3 of the average of all the responses, **rounded downward** to the nearest integer, wins.”
- Solve the modified game using iterated weak dominance.

4. (a) Solve for the Nash equilibrium of the following variant on Matching Pennies, and explain why the equilibrium (expected) payoff for each player equals zero:

		<u>Player II</u>	
		L	R
<u>Player I</u>	T	2 , -2	-2 , 2
	B	-1 , 1	1 , -1

- (b) Solve for the Nash equilibrium of the following variant on Matching Pennies, and explain why the equilibrium (expected) payoff for each player is not zero:

		<u>Player II</u>	
		L	R
<u>Player I</u>	T	2 , -2	-1 , 1
	B	-1 , 1	1 , -1

- (c) Solve for all of the Nash equilibria of the following variant on Battle of the Sexes, and compare with the equilibria of the original version of the game we discussed in class. Explain why some of the equilibria change and some of the equilibria stay the same.

		<u>F</u>	
		Boxing	Ballet
<u>M</u>	Boxing	3 , 1	0 , 0
	Ballet	0 , 0	1 , 3

5. Gibbons, page 50, problem 1.8.
6. Gibbons, page 51, problem 1.13.

7. Show that the following two-player game has a unique Nash equilibrium.
 (*Hint:* Show that it has a unique pure-strategy equilibrium; then show that player 1, say, cannot put strictly positive weight on both T and M; then show that player 1 cannot put strictly positive weight on both T and B, but zero weight on M; and finally show that player 1 cannot put strictly positive weight on both M and B, but zero weight on T.)

	L	C	R
T	1, -2	-2, 1	0, 0
M	-2, 1	1, -2	0, 0
B	0, 0	0, 0	1, 1

8. Determine *all* of the Nash equilibria of the following two-player game:

	L	C	R
T	2, 1	0, 0	0, 0
M	0, 0	2, 1	0, 3
B	0, 0	0, 3	2, 1

9. Consider the following *three-person*, zero-sum game, in which each player chooses between strategy A and strategy B. (The correct interpretation is as follows: the row indicates which strategy was chosen by player I; the column indicates which strategy was chosen by player II. If player III chooses strategy A, then the three players' payoffs are given by the first matrix; if player III chooses strategy B, then the three players' payoffs are given by the second matrix.)
- (a) Find the two Nash equilibria in pure strategies.
- (b) Find the Nash equilibrium in mixed strategies.

		II	
		A	B
I	A	2, 2, -4	0, 0, 0
	B	0, 0, 0	4, 4, -8

Player III chooses A

		II	
		A	B
I	A	3, 3, -6	0, 0, 0
	B	0, 0, 0	1, 1, -2

Player III chooses B

10. A Florida Voting Model

Consider a game played between two political candidates. Locations on the political spectrum are denoted by the variable x , where $x \in [0,1]$. Voters are uniformly distributed on $[0,1]$. The two candidates, identified as R and D, simultaneously and independently locate themselves at points $y_R \in [0,1]$ and $y_D \in [0,1]$, respectively.

If the two candidates select different locations, then a voter at location x votes for the candidate that she is located closer to.

If the two candidates select the same location, then the voters attempt to divide themselves 50-50 between casting votes for the two candidates. However, a small fraction of the votes for candidate D are not counted, so that candidate R actually receives 50.1% of the vote count and candidate D actually receives 49.9% of the vote count in this situation.

A candidate receives a payoff of +1 if he obtains greater than 50% of the vote count, receives a payoff of -1 if he obtains less than 50% of the vote count, and receives a payoff of 0 if he obtains exactly 50% of the vote count.

- (a) Concisely specify the candidates' strategy spaces and payoff functions.
- (b) Solve the game for: (i) *all* strictly-dominated strategies; and (ii) *all* Nash equilibria. In the course of answering, be sure to explicitly state the candidates' equilibrium strategies and to determine the candidates' equilibrium payoffs.
- (c) Compare your findings of strictly-dominated strategies and Nash equilibria with the corresponding results from the standard (non-Florida) voting game.

11. Consider the following static game. There are two players. Each player simultaneously and independently selects a strategy from a discrete set $\{0,1,2,\dots,10\}$. The payoff functions are given by:

$$U_1(s_1, s_2) = s_1 + s_2 + s_1s_2 - s_1^2$$

$$U_2(s_1, s_2) = s_1 + s_2 + s_1s_2 - 2s_2^2$$

(Note that U_1 and U_2 are different.)

Is the game solvable by iterated elimination of strictly dominated strategies? If so, solve it by iterated elimination of strictly dominated strategies. If not, indicate why it is not solvable by iterated elimination of strictly dominated strategies. In either event, determine *all* of the Nash equilibria of the game, and fully justify your answer.

12. A Vaccination Game

Six unvaccinated adults share a house, in a large state with an appallingly low vaccination rate. Their commonly-held belief is that the house is their main potential point of exposure to getting a disease. They believe that: any one of them who is vaccinated will not get the disease (probability of getting the disease is zero); and if one of them goes unvaccinated, the probability of that person getting the disease is proportional to the number of members of their group who are unvaccinated. Specifically, if a person is the only member of the group to go unvaccinated, then the probability that she gets the disease is 1/6. If she and one other member

of the group are not vaccinated (i.e., two in all) then the probability that they get the disease is $2/6$. If she and two other members of the group are not vaccinated (i.e., three in all) then the probability that they get the disease is $3/6$, etc.

They play a static game in which each player has two strategies: getting vaccinated (V) or going unvaccinated (U). Their payoffs are determined as follows: We denote the members of the group as players 1, 2, 3, 4, 5 and 6, respectively. The vaccination cost of player 1 is $c_1 = 1$, the vaccination cost of player 2 is $c_2 = 2$, ..., and the vaccination cost of player 6 is $c_6 = 6$. (More concisely, $c_i = i$, for all i ($1 \leq i \leq 6$)). The payoff (excluding the vaccination cost) to any player is 6, if she does not catch the disease, and it equals 0, if she catches the disease. Every player is an expected utility maximizer. Hence, the expected payoff, U_i , to any player i is given by $U_i = 6(1 - p_i) - c_i$, where p_i is the probability that player i catches the disease, following the previous paragraph (and is zero, if the player gets vaccinated) and c_i is the cost of the vaccination (and is zero, if the player goes unvaccinated).

The players move simultaneously and independently, and each player's objective is to maximize her own payoff.

- (a) Explain concisely and rigorously whether or not it is a Nash equilibrium for players 1, 2, 3 and 4 to get vaccinated and for players 5 and 6 to go unvaccinated.
- (b) Explain concisely and rigorously whether or not it is a Nash equilibrium for players 1, 2, and 3 to get vaccinated and for players 4, 5 and 6 to go unvaccinated.
- (c) Which players in this game, if any, have strictly or weakly dominated strategies? Explain your answers carefully, including whether any domination is strict or weak.

In parts (d) and (e), you should assume that the game has a structure that justifies applying iterated elimination of weakly dominated strategies (IEWDS), as well as iterated elimination of strictly dominated strategies (IESDS).

- (d) If we delete all strictly and weakly dominated strategies for all players from the game, which players in the resulting game, if any, now have (iteratively) strictly or weakly dominated strategies? Explain carefully, including whether any domination is strict or weak.
- (e) Find **all** of the Nash equilibria of this game, including both pure-strategy and mixed-strategy Nash equilibria. Provide a careful and complete justification for your answer.
- (f) What is the socially optimal outcome of this game? (That is, describe the outcome that would be determined by a social planner who cares only about the sum of the utilities of the six players who share this house and does not care about the outside world.) Explain concisely why the social optimum differs from the Nash equilibrium outcome in this game.