Economics 603 Professor Lawrence Ausubel
Fall 2016–2017

Final Exam

Answer all four questions, each in a *different* exam book.
Write your student number on each exam book and include your name on the book for Question 1.
Please return this question sheet at the same time that you turn in the exam books.
Write legibly, show your work, and justify your responses.

1. [20 points] **A static Cournot game with different cost functions**

(a) Consider the static Cournot game with two profit-maximizing firms, denoted A and B. Firm A's total cost function is given by $C_A(q_A) = (q_A)^2$. Firm B's total cost function is $C_B(q_B) = q_B$. The firms simultaneously and independently choose non-negative quantities, q_A and q_B. The price is determined by the inverse demand function $P(Q) = 20 - Q$, where $Q = q_A + q_B$.

Solve for the Nash equilibrium of this game.

(b) State, precisely, the hypotheses and conclusion of the Brouwer Fixed Point Theorem.

Now consider a more general Cournot model where firm A's total cost function is given by $C_A(q_A) = (q_A)^\alpha$ and firm B's total cost function is $C_B(q_B) = (q_B)^\beta$, where $\alpha \geq 1$ and $\beta \geq 1$. The price is determined by the inverse demand function $P(Q) = a - bQ$, where $Q = q_A + q_B$ and $a > b > 0$. Use the Brouwer Fixed Point Theorem to argue carefully that there always exists a Nash equilibrium of this more general Cournot game.

2. [25 points] **Auctions with three bidders and a reserve price**

A single item is being auctioned. Each of the auctions considered in this question ***has a reserve price of ¼***. The lowest nonzero bid that can be submitted is ¼; and the auction outcome is determined as if there were an additional "phantom" or "dummy" bidder who had submitted a bid of ¼. Thus, the high bidder wins only if her bid is at least ¼. (If instead, all real bidders bid zero, then the real bidders win nothing and the seller is paid nothing.) In addition, in any determination of payments, the second-highest bid is treated as always being at least ¼. Ties are resolved with the toss of a fair coin.

There are three risk-neutral bidders whose valuations are private information and are drawn from i.i.d. random variables that are uniformly distributed on the interval [0,1]. Each bidder knows her own valuation but only knows the distribution of her opponents' valuations. The bidders simultaneously and independently submit sealed bids in the interval [¼ ,1] (or the number zero).

(a) Determine an equilibrium of the second-price auction with a reserve price of ¼.

(b) Now consider the first-price auction with a reserve price of ¼. In any symmetric equilibrium, determine $P_i(v_i)$, the expected probability that the bidder of valuation type v_i wins the item (for each $v_i \in [0,1]$).

(c) Using part (b), solve for a Bayesian-Nash equilibrium of the first-price auction with a reserve price of ¼.

(d) Provide a complete statement of the Revenue Equivalence Theorem, and explain carefully whether or not the Revenue Equivalence Theorem applies to a comparison of the second-price auction and the first-price auction with the same reserve price.

3. [25 points] **An infinitely-repeated game**

Let **G** denote the following static game of complete information:

		II: L	C	R
I	T	10,10	4,12	1,1
	M	13,4	7,7	1,1
	B	4,4	4,4	0,0

(a) Find the Nash equilibrium of **G** and argue, very concisely but rigorously, that it is the unique Nash equilibrium of **G**.

(b) Now consider the infinitely-repeated version of **G** where players have the common discount factor of $\delta \in (0,1)$. Fully specify the trigger strategies that will maximize the sum of the expected payoffs of the two players, and solve for the critical discount factor $\hat{\delta}$ such that these trigger strategies form a subgame perfect equilibrium whenever $\delta \in (\hat{\delta},1)$.

(c) Fully specify the strategies for the "maximally-collusive" equilibrium that will maximize the sum of the expected payoffs of the two players and in which the punishment is carried out for exactly one period following any deviation, and solve for the critical discount factor $\hat{\delta}$ such that these trigger strategies form a subgame perfect equilibrium whenever $\delta \in (\hat{\delta},1)$.

(d) Are the critical discount factors $\hat{\delta}$ of parts (b) and (c) the same or different? Explain.

4. [30 points] **Alternating-offer bargaining for a piece of land**

Consider the following dynamic bargaining games of complete information between two sellers and one buyer. Each of the sellers owns a plot of land and values his land at 0. The value for the Buyer if he gets both plots of land is assumed to be 1, but the Buyer's value is 0 if he only gets one plot. Let p_t, q_t be the prices for the land of Seller 1 and Seller 2, respectively, proposed in period *t*. Assume that all of the players have the common discount factor $\delta \in (0,1)$.

A contingent contract of price p is an agreement between the Buyer and Seller 1, such that Seller 1 will sell his property to the Buyer at price p ***at the time*** when the Buyer and Seller 2 reach an agreement to trade Seller 2's property. Furthermore, under the contingent contract, if the Buyer does not reach agreement with Seller 2, then the trade with Seller 1 will not occur and no payment will be made.

Parts (a) and (b) will use contingent contracts with Seller 1, while parts (c) and (d) will use non-contingent sales to Seller 1.

(a) Consider the case where the Buyer and Seller 1 have already agreed upon a contingent contract with price p. Now the Buyer and Seller 2 play an infinite-period alternating-offer bargaining game, where the Buyer makes the first offer. That is, in the first period, the Buyer offers q_1 to Seller 2, who then decides to accept or to reject. If Seller 2 accepts, then the payoffs to the Buyer, Seller 1 and Seller 2 are $(1 - p - q_1, p, q_1)$. If Seller 2 rejects, then in the second period, he makes a counteroffer q_2 to the Buyer, who then decides to accept or to reject. If the Buyer accepts, then the payoffs are $(\delta(1 - p - q_2), \delta p, \delta q_2)$. If the Buyer rejects, then he makes a counteroffer in period 3, and so on. Since the contract with Seller 1 was contingent, the payoffs if the Buyer and Seller 2 fail to reach agreement are (0, 0, 0). Fully specify the unique subgame perfect equilibrium of this game and solve for the equilibrium payoffs of all three players. [*Note: In Question 4, you are not expected to argue why this equilibrium is unique.*]

(b) Now consider the bargaining between the Buyer and Seller 1 over the price of the contingent contract. In the first period, the Buyer offers a contingent contract of price p_1 to Seller 1, who then decides to accept or to reject. If Seller 1 accepts, then given the contingent contract of price p_1, the Buyer and Seller 2 play an infinite-period alternating-offer bargaining game where the Buyer makes the first offer immediately. That is, the game continues exactly as described in part (a). If Seller 1 rejects, then in the second period, Seller 1 offers a contingent contract of p_2 to the Buyer, who then chooses to accept or to reject. If the Buyer accepts, then again, given the contingent contract of price p_2, the Buyer and Seller 2 play an infinite-period alternating-offer bargaining game where the Buyer makes the first offer at the current time (meaning that the payoffs of the game described in part (a) would already be discounted by δ). If the Buyer rejects, then the game continues with the Buyer making a contingent contract offer of price p_3 in period 3, and so on. Fully specify the unique subgame perfect equilibrium of this game and solve for the equilibrium payoffs of all three players.

(c) Consider now the case where the Buyer has already bought Seller 1's land at price p and paid in full. But notice this plot of land is worth zero to the Buyer unless he buys Seller 2's plot as well. Now the Buyer and Seller 2 play an infinite-period alternating-offer bargaining game where the Buyer makes the first offer. The bargaining proceeds similarly to its description in part (a). Notice that the payoffs if Seller 2 accepts in period one are still $(1-p-q_1, p, q_1)$, but the payoff if the Buyer accepts in period 2 becomes $(-p+\delta(1-q_2), p, \delta q_2)$, as the price p was paid at the beginning of the game and hence will not be subject to discounting. Moreover, since the sale with Seller 1 was ***not*** contingent, the payoffs if the Buyer and Seller 2 fail to reach agreement anytime in the game are now $(-p, p, 0)$. Fully specify the unique subgame perfect equilibrium of this game and solve for the equilibrium payoffs of all three players.

(d) Finally, consider the bargaining game between the Buyer and Seller 1 without contingent contracts. The bargaining proceeds similarly as described in part (b). However, since the sale with Seller 1 was ***not*** contingent, if Seller 1 accepts p_1 in period one, the Buyer must pay Seller 1 an amount of p_1 in period one. Then the Buyer and Seller 2 play an infinite-period alternating-offer bargaining game where the Buyer makes the first offer immediately. Similarly, if the Buyer accepts p_2 in period two, he pays p_2 to Seller 1 in period two and he begins bargaining with Seller 2 immediately. That is, once the Buyer and Seller 1 have reached agreement, the rest of the game continues exactly as described in part (c). Fully specify the unique subgame perfect equilibrium of this game and solve for the equilibrium payoffs of all three players.

(e) Briefly, but intuitively, explain any difference between the equilibrium payoffs in parts (b) and (d) — or, if the answers are the same, explain why they are the same.