Question 2 Final Exam Economics 603 Fall 2008-2009

[25 points] **An infinitely-repeated game**

Consider the infinitely-repeated game, with a discount factor of $\delta \in (0,1)$, based on the following stage game:

| | | Player 2 | | |
|---|---|---|---|---|
| | | L | C | R |
| | U | 10, 10 | 2, 11 | 0, 13 |
| Player 1 | M | 11, 2 | 6, 7 | 0, −1 |
| | B | 12, −2 | 0, −5 | 3, 3 |

(a) Find all pure-strategy Nash equilibria of the stage game.

(b) Fully specify a pair of pure strategies in the infinitely-repeated game which, if followed, yield (10,10) as the average payoff.

(c) Find the critical discount factor to support the strategies of part (b) as a subgame-perfect equilibrium of the infinitely-repeated game.

(d) Is it possible to support (10,10) as an average payoff using a trigger-strategy equilibrium ***with a pure-strategy punishment*** when $\delta = 0.275$? If it is possible, carefully specify the equilibrium strategies. If it is impossible, briefly explain why.

(e) Is it possible to support (10,10) as an average payoff using a trigger-strategy equilibrium ***with a mixed-strategy punishment*** when $\delta = 0.275$? If it is possible, carefully specify the equilibrium strategies. If it is impossible, briefly explain why.

Solution:

a) S1 = {U,M,B}, S2 = {L,C,R} Pure NE = { (M,C) , (B,R)}

b) Any NE can be used as a punishment. For example:

| Strategy of Player 1 | | | Strategy of Player 1 | |
|---|---|---|---|---|
| Period 1 | Play U | | Period 1 | Play L |
| Period t | Play U if nobody deviated in previous periods | | Period 2 | Play L if nobody deviated in the previous periods |
| | Play M otherwise | | | Play C otherwise |

c) Critical discount factor: (clearly Player 2 has stronger incentives to deviate)

$$U_2^{EQ} = \frac{10}{1-\delta} \geq 13 + \frac{7\delta}{1-\delta} \qquad \underline{\delta} = 1/2$$

$$\delta \geq 1/2$$

Thus, for any $\delta \in [1/2, 1)$ the specified pair of strategies of part (c) constitute a subgame-perfect equilibrium.

e) Clearly, the trigger strategy of part (c) can be used when $\delta = 0.275$ . To support trigger-strategy equilibrium we need a stronger punishment for deviation. For example: another NE with payoffs (3,3)

| Strategy of Player 1 | | | Strategy of Player 1 | |
|---|---|---|---|---|
| Period 1 | Play U | | Period 1 | Play L |
| Period t | Play U if nobody deviated in previous periods | | Period 2 | Play L if nobody deviated in the previous periods |
| | Play B otherwise | | | Play R otherwise |

Clearly, Player 2 has stronger incentives to deviate. Thus, the critical discount factor is:

$$U_2^{EQ} = \frac{10}{1-\delta} \geq 13 + \frac{3\delta}{1-\delta} \qquad \underline{\delta} = 0.3$$

$$\delta \geq 3/10$$

Thus, we can support (10, 10) with trigger-strategy equilibrium which includes pure-strategy punishment when $\delta = 0.275$ .

f) \. We need another NE which we can use as a punishment.

Mixed equilibrium:

| | | Player 2 | | |
|---|---|---|---|---|
| | | L | C | R |
| | U | 10,10 | 2,11 | 0,13 |
| Player 1 | M | 11,2 | 6,7 | 0,-1 |
| | B | 12,-2 | 0,-5 | 3,3 |

1) The mixed strategy: 0.5 M + 0.5 B strictly dominates U
2) The mixed strategy: 0.5 C + 0.5 R strictly dominates L

Than, it is not hard to see that the mixed equilibrium is as follows:
Mixed NE = { (1/2 M + 1/2 B , 1/3 C + 2/3 R)} with payoff (2,1).

| | | | | |
|---|---|---|---|---|
| Strategy of Player 1 | | | Strategy of Player 1 | |
| Period 1 | Play U | | Period 1 | Play L |
| Period t | Play U if nobody deviated in previous periods | | Period 2 | Play L if nobody deviated in the previous periods |
| | Play (1/2 M + 1/2 B) otherwise | | | Play (1/3 C + 1/3 R) otherwise |

Clearly, Thus, the critical discount factor is:

$$U_1^{EQ} = \frac{10}{1-\delta} \geq 12 + \frac{2\delta}{1-\delta} \qquad U_2^{EQ} = \frac{10}{1-\delta} \geq 13 + \frac{1\delta}{1-\delta} \qquad \underline{\delta} = 0.25$$

$$\delta \geq 2/10 \qquad \delta \geq 1/4$$

Thus, for any $\delta \in [0.25, 1)$ the specified pair of strategies of part (c) constitute a subgame-perfect equilibrium.

Question 3 Final Exam Economics 603 Fall 2008-2009

[25 points] **Cournot oligopoly with fixed costs**

There are four symmetric firms in a market. The inverse demand function for this market is given by $P(Q)=11-Q$ , where $Q=\sum_{i=1}^{4} q_i$ is total quantity. Each firm *i* simultaneously and independently chooses a quantity $q_i$ (i.e. as in a standard Cournot model), but each firm has the following cost function: $C(q)=\begin{cases} 0 & q=0 \\ q+F & q>0 \end{cases}$ , where $F$ is a parameter of the model.

(a) Argue why is it possible to have equilibria with different number of firms which produce at positive level of output for the same values of *F*.

(b) Solve for all Nash equilibria in pure strategies for all possible $F\in[2,5]$ .

(c) Consider a two-stage dynamic game where in the first stage firms decide whether or not to enter, and in the second stage (after observing the entry decision of each competitor) firms compete in quantities. If t $C(q)=q$ he fir $F\in[2,5]$ m decides to enter, it must pay *F*. The cost function of production is . For all, find all SPE in pure strategies. [Hint: Use the equilibria of part (b) and argue which of them are SPE.]

(d) Consider the dynamic game of part (c) with the following modification. After entering the firms **do not obs** $F\in[2,5]$ **erve** the entry decision of their opponents. Explain how to analyze this game and, for all, find all SPE in pure strategies.

**Solution:**

(a) In all equilibria active firms (those with positive level of production) will be producing corresponding Cournot quantities . Depending on the number of active firms this quantity is different. For the same value of fixed cost the number of active firms can be different. The fixed cost will prevent inactive players from entering.

(b) There are 5 possible types for equilibria in this game which can be described by the number of active firms in a market.

General solution for the Cournot model with N active ( $q_i>0$ ) firms.

$$q^*=\frac{A-c}{N+1} \qquad Q^*=\frac{N(A-c)}{N+1} \qquad P^*-c=\frac{A-c}{N+1} \qquad \pi^*=\left(\frac{A-C}{N+1}\right)^2$$

**Case 1**: All firms are in a market (N=4)

$$q^*=\frac{10}{5}=2 \qquad Q^*=\frac{4*10}{5}=8 \qquad P^*-c=\frac{10}{5}=2 \qquad \pi^*=\left(\frac{10}{5}\right)^2=4$$

Thus, for $F\in[2,4]$ there is an equilibrium of this type. $NE=[(2,2,2,2)]\forall F\in[2,4]$. For $F\in(4,5]$ there are no equilibrium of this type.

**Case 2**: Three firms are active. (N = 3)

$$q^*=\frac{10}{4}=2.5 \qquad Q^*=\frac{3*10}{4}=7.5 \qquad P^*-c=\frac{10}{4}=2.5 \qquad \pi^*=\left(\frac{10}{4}\right)^2=6.25$$

By construction, the firms with positive production will not deviate $\forall F\in[2,5]$. The optimal deviation of the firm with zero production level is to act as a monopolist on the demand residual.

$$q^d : Max\,(A-c-Q^*-q^d)q^d-F$$

$q^d=1.25 \qquad Q=8.75 \qquad P-c=1.25 \qquad \pi^d=1.25^2-F=1.5625-F<0 \quad \forall F\in[2,5]$ Thus, the inactive firm will not deviate. Consequently, there are four equilibria for $\forall F\in[2,5]$
$NE=[(2.5,2.5,2.5,0),\ (2.5,2.5,0,2.5),\ (2.5,0,2.5,2.5),\ (0,2.5,2.5,2.5)]$

**Case 3**: Two firms are active (N =2)

$$q^*=\frac{10}{3}=10/3 \qquad Q^*=\frac{2*10}{3}=20/3 \qquad P^*-c=\frac{10}{3}=10/3 \qquad \pi^*=\left(\frac{10}{3}\right)^2=100/9$$

By construction, the firms with positive production will not deviate $\forall F\in[2,5]$. The optimal deviation of the firm with zero production level is to act as a monopolist on the demand residual.

$$q^d : Max\,(A-c-Q^*-q^d)q^d-F$$

$$q^d=5/3 \qquad Q=25/3 \qquad P-c=5/3 \qquad \pi^d=(5/3)^2-F=25/9-F$$

Thus, the inactive firm will not deviate only in case $F\in[25/9,5]$. Consequently, there are six equilibria for $\forall F\in[25/9,5]$
$NE=[(10/3,10/3,0,0),\ (10/3,0,10/3,0),\ (10/3,0,0,10/3),\ldots$
$\ldots(0,10/3,10/3,0),\ (0,10/3,0,10/3),\ (0,0,10/3,10/3)]$

**Case 4**: One firm is active (N=1). Monopoly.

$$q^*=\frac{10}{2}=5 \qquad Q^*=\frac{1*10}{2}=5 \qquad P^*-c=\frac{10}{2}=5 \qquad \pi^*=\left(\frac{10}{2}\right)^2=25$$

By construction, the firm with positive production will not deviate $\forall F\in[2,5]$. The optimal deviation of the firm with zero production level is to act as a monopolist on the demand residual.

$$q^d : Max\,(A-c-Q^*-q^d)q^d-F$$

$q^d = 2.5 \quad Q = 7.5 \quad P - c = 2.5 \quad \pi^d = (2.5)^2 - F = 6.25 - F > 0 \quad \forall F \in [2,5]$

Thus, profit from deviation is higher than 0 for any possible F. No equilibrium of this type.

**Case 5**: Zero active firms.

Then any firm can deviate and earn monopoly profit. No such equilibrium.

**To sum up**:

| Range for F | Number of equilibria |
|---|---|
| $F \in [2, 25/9)$ | 5 |
| $F \in [25/9, 4]$ | 11 |
| $F \in (4, 5]$ | 10 |

(c) If $F \in [2,4)$ then there is a unique symmetric SPE.

Equilibrium strategy:
Stage 1: Enter
Stage 2: (N is a total number of players who played enter in the first stage)

$$q_i^*(N) = \begin{cases} 2 & N=4 \\ 2.5 & N=3 \\ 10/3 & N=2 \\ 5 & N=1 \end{cases}$$

Equilibrium payoffs: $\pi_i^* = 4 - F \geq 0 \quad \forall F \in [2,4)$

If $F = 4$ then there are five SPE. One of them is the same as the previous one and four asymmetric ones where one firm stays out. The strategy for the second stage is the same.

For $F > 4$ there are four asymmetric SPE where one firm stays out.

(d) If firms do not observe the entry decisions by their opponents, the structure of the game is exactly the same as in (b). There are no proper subgames except for the whole game. The set of SPE will be the same as the set of NE. All equilibria can be found in part (b).