Demand Reduction and Inefficiency in Multi-Unit Auctions

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12 July 2002

(first draft: 8 November 1995)

Abstract

Auctions typically involve the sale of many related goods. Treasury, spectrum and electricity auctions are examples. In auctions where bidders pay the market-clearing price for items won, large bidders have an incentive to reduce demand in order to pay less for their winnings. This incentive creates an inefficiency in multiple-item auctions. Large bidders reduce demand for additional items and so sometimes lose to smaller bidders with lower values. We demonstrate this inefficiency in an auction model which allows interdependent values. We also establish that the ranking of the uniform-price and pay-as-bid auctions is ambiguous in both revenue and efficiency terms. Bidding behavior in spectrum auctions, electricity auctions, and experiments highlights the empirical importance of demand reduction.

JEL No.: D44 (Auctions)
Keywords: Auctions, Multi-Unit Auctions, Spectrum Auctions, Treasury Auctions, Electricity Auctions

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*The authors gratefully acknowledge the support of the National Science Foundation. We appreciate valuable comments from Preston McAfee, three anonymous referees, and participants at numerous conferences and seminars.

One of the preeminent justifications for auctioning public resources is to attain allocative efficiency. For example, Vice President Al Gore opened the December 1994 Broadband PCS spectrum auction proclaiming, "Now we're using the auctions to put licenses in the hands of those who value them the most."[1] Given the emphasis that policymakers place on efficiency, surprisingly little is known by economists about the efficiency properties of various auction designs for multiple items. Most of the conventional wisdom comes by analogy from single-item auctions. We know that the second-price auction and the English auction induce buyers to bid sincerely, implying efficient outcomes (William Vickrey 1961). Under a first-price auction, buyers shade their bids below their values, but efficiency is still possible when symmetric bidders employ symmetric strategies.

However, in environments where bidders desire multiple items, general results, beyond those in Vickrey's original paper, are not well understood. This observation is clearest in the context of U.S. Treasury auctions, where there has been a longstanding debate between two alternatives. The traditional format used for the sale of Treasury securities has been the *pay-as-bid auction*: bidders each submit bids for various quantities at various prices, the auctioneer determines the market-clearing price, all bids exceeding the market-clearing price are accepted, and bidders pay their winning bids. Milton Friedman (1960) proposed the *uniform-price auction*: bidders each submit bids for various quantities at various prices, the auctioneer determines the market-clearing price, all bids exceeding the market-clearing price are accepted, and bidders pay the market-clearing price for all units won, rather than their actual bids.

Most public debate about the relative merits of these two alternatives has been confused by an imperfect analogy between single-unit and multi-unit auctions. Academics and policymakers, alike, have observed that the pay-as-bid auction can be viewed as a multi-unit extension of the first-price auction, and have asserted that the uniform-price auction is best seen as a multi-unit extension of the second-price auction. This flawed analogy has led otherwise-astute economists to incorrectly posit that the uniform-price auction inherits the same attractive truth-telling and efficiency attributes as the second-price sealed-bid auction. It has also led observers to wrongly infer that the uniform-price auction ought to—as a general theoretical matter—generate greater expected seller revenues than a pay-as-bid auction.

In the second-price auction of a single item with private values, bidding one's own true value is a weakly-dominant strategy. The notion that sincere bidding does not extend to a uniform-price auction where bidders desire multiple units originates in the seminal work of Vickrey (1961). Nevertheless, what might be called the "uniform-price auction fallacy" is still often made, most conspicuously in discussions

[1] In the *Omnibus Budget Reconciliation Act of 1993*, which authorized spectrum auctions, the U.S. Congress established the "efficient and intensive use of the electromagnetic spectrum" as a primary objective of U.S spectrum auctions (47 U.S.C. § 309(j)(3)(D)).

of the U.S. Treasury auction. In the *Wall Street Journal* (August 28, 1991), Friedman appeared to assert that with a uniform-price auction one simply bids one's reservation value: "A [uniform-price] auction proceeds precisely as a [pay-as-bid auction] with one crucial exception: All successful bidders pay the same price, the cut-off price. An apparently minor change, yet it has the major consequence that no one is deterred from bidding by fear of being stuck with an excessively high price. You do not have to be a specialist. You need only know the maximum amount you are willing to pay for different quantities." Merton Miller, in an interview with *The New York Times* (September 15, 1991, 3:13) also appeared to believe that there was no incentive for bid shading in the uniform-price auction: "All of that is eliminated if you use the [uniform-price] auction. You just bid what you think it's worth." The *Joint Report on the Government Securities Market* (1992, p. B-21), jointly signed by the Treasury Department, the Securities and Exchange Commission, and the Federal Reserve Board, stated: "Moving to a uniform-price award method permits bidding at the auction to reflect the true nature of investor preferences … . In the case envisioned by Friedman, uniform-price awards would make the auction demand curve identical to the secondary market demand curve."

One of the objectives of our paper is to clear the air of the uniform-price auction fallacy. We demonstrate in generality that a bidder who desires more than one unit in a uniform-price auction has an incentive to shade her bid. Moreover, we show that this demand reduction necessarily leads to inefficiency: *every* equilibrium of the uniform-price auction is ex post inefficient with positive probability.

The extent of demand reduction and the resulting inefficiency depends on the presence of large bidders, who have an ability to exercise market power. Even in Treasury auctions, where the number of participants is large, the top five bidders typically purchase nearly one-half of the issue (Malvey and Archibald 1998). Electricity and spectrum markets exhibit even higher levels of concentration. As a result, demand reduction (supply reduction in the case of electricity auctions) is not just of theoretical interest, but is of great practical importance, both in terms of auction design and bidding strategy.

A second objective of our paper is to settle the ranking of the pay-as-bid and uniform-price auction. For over 40 years, it has been an open question—both theoretically and empirically—whether the pay-as-bid format or the uniform-price format would yield greater revenues in Treasury auctions. Friedman (1960) conjectured that the uniform-price auction would dominate the pay-as-bid auction in revenues. This and the 1991 Salomon Brothers scandal led the U.S. Treasury to experiment with the uniform-price rule for selling two-year and five-year notes from 1992 to 1998 and to switch entirely to the uniform-price format in November 1998. However, the Treasury experiment yielded inconclusive results: "In a direct comparison of the impact on revenues between the two techniques, the data show a small increase in revenues to the Treasury under the uniform-price technique, but the difference is not statistically significant." (Malvey and Archibald 1998, p. 14; see also Malvey, Archibald and Flynn 1996 and Reinhart and Belzer 1996).[2] Using Turkish data, Hortascu (2002) found that the pay-as-bid auction produced more revenue, ex post, than would have the uniform-price auction, but he failed to reject ex ante expected revenue equality. The question also has spawned an extensive literature of laboratory experiments, which has tended to slightly favor the uniform-price auction, except when bidders' demand curves are sufficiently steep (Smith 1967, 1982).

In our paper, we compare the two auction formats. Considering the objectives of efficiency and revenue maximization, we find that the ranking is inherently ambiguous.[3] We construct reasonable specifications of demand where the pay-as-bid auction dominates the uniform-price auction both on expected gains from trade and expected seller revenues. We also construct equally reasonable specifications of demand where the reverse ranking holds. Thus, if the seller is constrained to select between the pay-as-bid and uniform-price auction, the choice ought to be viewed as an empirical question.

The theorems of our paper are stated formally for static multi-unit auctions where bidders submit bid curves, and so the theorems are most obviously applicable to sealed-bid auctions such as those for Treasury bills or electricity. However, most of our results can be adapted to any auction context where equilibria possess a uniform-price character. For example, in the simultaneous ascending auctions used for spectrum licenses, there is a strong tendency toward arbitrage of the prices for identical items. Indeed, in the FCC's Nationwide Narrowband Auction of July 1994, similar licenses were on average priced within 0.3 percent of the mean price for that category of license, and the five most desirable licenses sold to three different bidders identically for $80 million each. Similarly, consider items that are sold through a sequence of (single-item) English auctions. The declining-price anomaly notwithstanding, there is a reasonable tendency toward intertemporal arbitrage of the prices for identical items, and so a variant on our Inefficiency Theorem should typically apply.

The intuition for bid shading and demand reduction in the uniform-price auction is as follows. When a bidder desires multiple units of the good being auctioned, there is a positive probability that her bid on a

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[2] Interestingly, in switching to uniform pricing, the Treasury was apparently motivated in part by an incorrect extension to the multi-unit setting of Milgrom and Weber's (1982) famous result that the second-price auction generates greater revenue than the first-price auction: "One of the basic results of auction theory is that under a certain set of assumptions the revenue to the seller will be greater with uniform-price auctions than with [pay-as-bid] auctions." (Malvey and Archibald 1998, p. 3)

[3] In important earlier work, Back and Zender (1993) in a pure common value setting demonstrated that revenues may be lower from the uniform-price auction than from a particular equilibrium of the pay-as-bid auction.

second or later unit will be pivotal, thus determining the price that the bidder pays on other units that she wins. Given this, she has an incentive to bid less than her true value on later units in order to reduce the price she will pay on the earlier units. With discrete goods, this intuition suggests that the bidder will bid her true value on her first unit demanded, but strictly less than her true value on all subsequent units. With divisible goods, this intuition suggests that a bidder's submitted demand curve will take on the qualitative features of a monopolist's marginal-revenue curve: at zero quantity the demand curve and the bid curve (marginal revenue curve) intersect, but at all positive quantities, the bid curve (marginal revenue curve) lies strictly below the true demand curve.

Our model allows for interdependent values. Each bidder's true demand may depend on information known by others. In such a setting, it is essential that the bidder condition her bid on the information revealed by winning a particular quantity of the good. We assume that winning a larger quantity of the good is worse news about the good's value, since winning more means that others do not value the good as highly as they might. As a result, the rational bidder shades her bid to avoid bidding above her conditional marginal value for the good, such bidding termed the Winner's Curse. Henceforth, we will refer to bid shading as bidding *below* the bidder's conditional marginal value for the good, rather than merely the shading that arises from winner's curse avoidance.

The Inefficiency Theorem relies not only on bid shading but on *differential bid shading*. This point is apparent from the standard first-price auction of a single item: every bidder shades her bid, but with symmetric bidders and in a symmetric equilibrium, higher bids still imply higher values. Inefficiency becomes mandatory when there are bidders with identical marginal values shading their bids by different amounts. Differential shading is present in the uniform-price auction, since the incentive to reduce demand increases with quantity. There is no bid shading on the first unit demanded, but increasing amounts of bid shading occur on subsequent units.

A simple example with independent private values illustrates the theorem. Suppose two identical items are sold to two bidders using a third-price auction—the uniform-price auction with two goods. Bidder 1 wants just a single item and has a value u. Bidder 2 wants up to two items and has a constant marginal value v for each item. The privately known values u and v are drawn independently from the uniform distribution on [0,1]. Each bidder submits two nonnegative bids. The seller ranks the four bids, awards the items to the two highest bids, and the winner(s) pay the third-highest bid for each item won.

As in the second-price auction for a single item, it is a weakly dominant strategy for both bidders to bid their true values for a first item. The only time a bidder's first bid determines the price is when it is the third-highest, in which case the bidder wins zero items and the price is irrelevant to the bidder. However, when the bid does not set the price, profits are maximized by making this bid compete favorably against all bids up to the bidder's true value—then the bidder wins whenever it is profitable. Hence, bidders 1 and 2 first bids are u and v, respectively. Bidder 1 bids zero for a second item, since her value is zero. The only remaining question is bidder 2's bid for the second item. Clearly, bidder 2 does not want to bid more than her value. By bidding $b \le v$, there are two possibilities: (1) if $b < u$, then bidder 2 wins one item and pays b, or (2) if $b > u$, then bidder 2 wins two items and pays u for each. Thus, bidder 2's expected profit $\pi(b)$ from the bid b is

$$\pi(b) = (v-b)(1-b) + \int_0^b 2(v-u)du = v-(1-v)b.$$

Hence, $b = 0$ maximizes $\pi(b)$ for all $v \in [0,1]$ and bidder 2 optimally bids 0 on the second item, regardless of how high her true valuation actually is. The demand reduction in this example is extreme. Both bidders win a single item and pay zero for the item won. The auction performs poorly in both revenue and efficiency terms.

The efficiency of the pay-as-bid auction may exceed that of the uniform-price auction for the following reason. In the pay-as-bid auction, the incentive to shade bids need not increase in quantity, since a bid for an additional unit in a pay-as-bid auction has no effect on the price that is paid for earlier units. So it is possible for bidders with similar marginal valuations at different quantities to be shading their bids by similar amounts, consistent with efficiency.

Engelbrecht-Wiggans (1988) and Maskin and Riley (1989) consider the case where bidders desire multiple units of the good. They show that the weak form of the Revenue Equivalence Theorem holds in an independent private value setting: each bidders surplus, and hence the seller's revenues, depends only on the allocation of the goods. Auctions that result in the same allocation of the goods, necessarily result in the same auction revenues. However, as we indicate here, the uniform-price, pay-as-bid, and Vickrey auctions generally assign the goods differently, so the strong form of revenue equivalence fails.

Wilson (1979) and subsequent authors (Back and Zender 1993, Wang and Zender 2002) develop the continuous methodology of "share auctions" that we exploit in the current paper. However, each of these papers assumes that bidders have pure common values, so that allocative efficiency is not an issue—every allocation is equally efficient. Back and Zender, as well as Wang and Zender, also address the issue of ranking the uniform-price and pay-as-bid auctions in terms of seller revenues for some specific functional forms. They face the methodological limitation of comparing one equilibrium (out of a multiplicity of equilibria) of the uniform-price auction with the equilibrium of the pay-as-bid auction. By contrast, our Inefficiency Theorem is a statement about the entire *set* of equilibria.

Several recent theoretical papers have begun to address the questions which we set forth above. Noussair (1995) and Engelbrecht-Wiggans and Kahn (1998) examine uniform-price auctions where each bidder desires up to two identical, indivisible items. They find that a bidder generally has an incentive to bid sincerely on her first item but to shade her bid on the second item. Engelbrecht-Wiggans and Kahn provide a construction which is suggestive of the inefficiency and revenue results we obtain below. They offer a particularly ingenious class of examples in which bidders bid zero on the second unit with probability one. Tenorio (1997) examines a model where each of two bidders desires up to three identical items, and each bidder is constrained to bid a single price for a quantity of either two or three. He finds that greater demand reduction occurs under a uniform-price auction rule than under a pay-as-bid rule.

Bolle (1997) addresses the efficiency question which we pose here. In an indivisible-goods framework restricted to independent private values, he simultaneously and independently concludes that equilibria of the uniform-price and pay-as-bid auctions are always inefficient.

Several recent experimental studies confirm the presence of demand reduction in uniform-price auctions. Kagel and Levin (2001) found substantial demand reduction with uniform pricing, regardless of whether the auction was static or dynamic. Similarly, List and Lucking-Reiley (2000) found demand reduction in Internet experiments with two units and two bidders. Engelbrecht-Wiggans, List, and Lucking-Reiley (1999) conducted Internet auctions with more than two bidders. Consistent with the theory, they found that demand reduction diminishes with competition, but does not vanish.

Our paper is organized as follows. Section 1 lays out the model and describes how the winner's curse generalizes when selling multiple items. Section 2 proves the Inefficiency Theorem for bidders with interdependent, constant marginal values (flat demands). Section 3 establishes the ambiguous ranking of the uniform-price and pay-as-bid auctions, both in efficiency and revenue terms. Section 4 proves the Inefficiency Theorem for bidders with downward-sloping demands. Section 5 provides examples useful in experiments and classroom exercises, where unique equilibria in weakly-dominant strategies are easily derived. Section 6 concludes, emphasizing the practical importance of demand reduction, as seen in spectrum and electricity auctions.

1 Preliminaries

We begin by specifying the multi-unit auction setting in which the inefficiency result will be developed. Except in Section 4, we will assume that each bidder has a constant marginal value for the good, up to a fixed capacity. While this assumption is restrictive, it simplifies the analysis yet includes most of the settings that have been analyzed in the earlier literature in which bidders possess unit demands. In particular, the "flat demands" model provides a generalization of Milgrom and Weber's (1982) model of an auction for a single object.

1.1 The Flat Demands Setting

The seller has a quantity, normalized to 1, of a divisible good to sell to n bidders. The seller's valuation for the good equals zero. Each bidder i can consume any quantity $q_i \in [0,\lambda_i]$, where $\lambda_i \in (0,1)$. Without loss of generality, we assume $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. We require that there is competition for each quantity of the good: $\lambda_2 + \lambda_3 + \ldots + \lambda_n \geq 1$. We can interpret q_i as bidder i's share of the total quantity, and λ_i as a quantity restriction. For example, in the U.S. Treasury auctions, a bidder's net long position, including both pre-auction trading and the auction award, cannot exceed 35 percent. The FCC spectrum auctions have had similar quantity restrictions.

Each bidder receives a private signal of valuation, before the time that bids are submitted. As is standard in the literature on games with asymmetric information, the signal will be referred to as the bidder's type. Let $t_i \in [0,1]$ be bidder i's type, let $t = (t_1,\ldots,t_n)$, and let $t_{-i} = t\backslash t_i = (t_1, \ldots , t_{i-1}, t_{i+1}, \ldots ,t_n)$. Types are drawn from the joint distribution F with strictly positive and finite density f on $(0,1)^n$. The distribution function F is commonly known to bidders, but the realization t_i is known only to bidder i.

Bidder i has a constant marginal value $V_i \in [0,1]$ for the good up to the capacity λ_i. A bidder i with marginal value V_i consuming Q_i and paying P_i has a payoff $u_i(V_i,Q_i,P_i) = Q_iV_i - P_i$, for $Q_i \in [0,\lambda_i]$. Values are interdependent in the sense that each bidder's value is allowed to depend on other bidders' types. The relationship $V_i = v_i(t)$ between types and values is assumed to satisfy:

VALUE MONOTONICITY. $v_i(t_i,t_{-i})$ is strictly increasing in t_i, $v_i(t_i,t_{-i})$ is weakly increasing in each component of t_{-i}, and $v_i(\cdot,\cdot)$ is continuous in all its arguments.

TYPES RANK VALUES. $t_i > t_j \Rightarrow v_i(t) > v_j(t)$.

The function $v_i(\cdot,\cdot)$ is also assumed to be common knowledge among bidders.

The model generalizes the independent private values model in two ways: values may depend on the private information of others, and a bidder's private information need not be independent of the private information of others. The Types Rank Values assumption deliberately excludes a pure common value model, since there, any assignment—and hence any auction without a reserve price—is efficient.

A critical element in the analysis of auctions for a single good is the first-order statistic. If X_i denotes the signal of bidder i and Y_i denotes the first-order statistic of the signals of all bidders other than bidder i, then bidder i receives the good in the efficient assignment if and only if $X_i \geq Y_i$. Moreover, bidder i's expected value from receiving the good is given by $v_i(x,y) = E[V_i \mid X_i = x,\ Y_i = y]$. Consequently, in a symmetric model, $v_i(x,x)$ is an equilibrium bidding strategy for each player in the second-price auction of a single good (Milgrom and Weber, 1982). In m-unit auctions where each bidder can win at most one unit, the m^{th}-order statistic serves the analogous role. However, for analyzing general multi-unit auctions, the order statistics by themselves are inadequate: the quantity won by a bidder confers additional information. In our first definition, we appropriately generalize the order-statistic notions to a multi-unit auction environment:

DEFINITION 1. *For any* $t \in [0,1]^n$, *let* $Q^*(t) \equiv (Q_1^*(t),\cdots,Q_n^*(t))$ *denote an efficient assignment of the good, that is, an assignment that maximizes* $\sum_{i=1}^n v_i(t)Q_i(t)$ *subject to* $Q_i(t) \in [0,\lambda_i]$, *for* $i=1,\ldots,n$, *and* $\sum_{i=1}^n Q_i(t) = 1$. *For any* $t_{-i} \in (0,1)^{n-1}$ *and* $q \in (0,\lambda_i]$, *define* $\tau_i^q(t_{-i}) = \inf\ \{t_i \in [0,1] : Q_i^*(t_i,t_{-i}) \geq q\}$. *Furthermore, let* $F_i^q(y \mid x) = Prob\{\tau_i^q(t_{-i}) \leq y \mid t_i = x\}$, *let* $f_i^q(y \mid x)$ *denote the associated density function, let* $w_i^q(x,y) = E[v_i(t_i,t_{-i}) \mid t_i = x,\ \tau_i^q(t_{-i}) = y]$, and (if defined) let $w_i^+(x,x) = \lim_{q\downarrow 0} w_i^q(x,x)$.

Translating the familiar environment and notation from auctions of single goods into the new multi-unit notation, we would have $\lambda_i = 1$, $\tau_i^1(t_{-i}) \equiv Y_i$, $F_i^1(y \mid x)$ is the probability that $Y_i \leq y$ conditional on $X_i = x$ and bidder i winning the good, $f_i^1(y \mid x)$ is the associated density, and $w_i^1(x,y) \equiv v_i(x,y)$.

Note that $Q^*(t)$ is uniquely defined on type vectors $t \in [0,1]^n$ with probability one, while $\tau_i^q(t_{-i})$ is uniquely defined for every $t_{-i} \in (0,1)^{n-1}$ and $q \in (0,\lambda_i]$. We will henceforth assume that the primitives of the model have been specified such that $F_i^q(y \mid x)$, $f_i^q(y \mid x)$ and $w_i^q(x,y)$, when needed, are mathematically well-defined functions, and such that $w_i^q(x,y)$ is continuous in (x,y).

1.2 The Generalized Winner's Curse

Essentially all of the previous auctions literature has made assumptions that imply the Winner's Curse, the notion that winning is "bad news": a bidder's expected value conditional on winning is less than or equal to her unconditional expected value. We find that an extension of the Winner's Curse to a multi-unit setting is an important regularity condition that needs to be satisfied.

Under the standard assumptions of the literature on auctions of a single good, recall that $v_i(x,x) \equiv E[V_i \mid X_i = x,\ Y_i = x] \leq E[V_i \mid X_i = x]$. In addition, $v_i(x,y)$ is strictly increasing in x and weakly increasing in y (Milgrom and Weber, 1982, p. 1100). The natural extension of the Winner's Curse to a multi-unit auction setting is to require $w_i^q(x,x)$ to be weakly decreasing in q, as well as to satisfy the same monotonicity properties as in the single-object environment.

DEFINITION 2. *A multi-unit auction environment exhibits the* ***Generalized Winner's Curse*** *if* $w_i^q(x,y)$ *is strictly increasing in x and weakly increasing in y, and if* $w_i^q(x,x)$ *is weakly decreasing in q.*

Thus, the Generalized Winner's Curse requires that winning a larger quantity is "worse news" than winning a smaller quantity. Note that it implies that $w_i^+(x,x)$ of Definition 1 is well defined.

1.3 Conventional Auctions

The seller uses a *conventional auction* to allocate the good. In a conventional auction, bidders submit bids, and the items are awarded to the highest bidders. The simplest formulation is for each bidder i to simultaneously and independently submit a *demand curve* $q_i(p)$:$[0,1]\rightarrow[0,\lambda_i]$. As always, the demand curve expresses the quantity that bidder i is willing to purchase at a price of p. We require the function, $q_i(\cdot)$, to be left continuous at $p = 1$, right continuous at all $p \in [0,1)$, and weakly decreasing. The market-clearing price, p^0, is determined from the highest rejected bid: $p^0 = \min\{p \mid \sum_i q_i(p) \le 1\}$.

The formal analysis is facilitated if our primitive notion of a bidder's strategy is instead a *bid function*, which is simply the inverse of a demand curve. To be precise, each bidder i submits a bid function $b_i(q)$:$[0,\lambda_i]\rightarrow[0,1]$, which is required to be right continuous at $q = 0$, left continuous at all $q \in (0,\lambda_i]$, and weakly decreasing. For the moment, we suppress the dependence of demand curves and bid functions on t_i. The seller inverts each bid function to obtain a demand curve and then determines the market-clearing price as above.[4]

Since each $q_i(\cdot)$ is weakly decreasing, and since (consistent with $V_i \in [0,1]$) the construction for inverting bid functions described in the previous footnote imposes that $q_i(0) = \lambda_i$ and $q_i(1) = 0$, observe that p^0 exists, p^0 is unique and $p^0 \in (0,1)$. If $\sum_i q_i(p^0) = 1$, then each bidder i is assigned a quantity of $q_i(p^0)$. If $\sum_i q_i(p^0) > 1$, then the aggregate demand curve is flat at p^0 and some bidders' demands at p^0 will need to be rationed.[5] All that remains to be specified is the pricing rule, P, and this is how the conventional auctions differ. This paper focuses on the two most common versions.

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[4] A demand curve is constructed from the bid function as follows. Let $\Gamma = \{(q, b_i(q)) \mid q \in [0,\lambda_i]\} \cup \{(0,1),(\lambda_i,0)\}$ denote the graph of $b_i(q)$ and the two additional points which say that, at a price of one, the bidder demands nothing, and at a price of zero, the bidder demands as much as possible. Take the closure of Γ, and then convexify the set in the direction of the range (i.e., fill in all the discontinuities of the demand curve), and call the result Γ'. Finally, we can define weakly-decreasing correspondence $\gamma_i(p) = \{q \mid (q,p) \in \Gamma'\}$, and define function $q_i(p)$ to be the selection from $\gamma_i(p)$ which is left continuous at $p = 1$ and right continuous at all $p \in [0,1)$.

[5] If there is just a single bidder whose demand curve is flat at p^0, then this bidder's quantity is reduced by $\sum_i q_i(p^0) - 1$. If there are multiple bidders with demand curves flat at p^0, then quantity is allocated by proportionate rationing. (For our purposes, the specific tie-breaking rule will not matter, since with probability one, there is at most a single bidder with flat demand at p^0.) In particular, define bidder i's incremental demand at p^0 as $\Delta_i(p^0) \equiv q_i(p^0) - \lim_{p\downarrow p^0} q_i(p)$. Then bidder i is awarded an amount $Q_i = q_i(p^0) - (\sum_i q_i(p^0) - 1)\Delta_i(p^0)/\sum_i \Delta_i(p^0)$.

UNIFORM-PRICE AUCTION. Each bidder i assigned Q_i pays the market-clearing price p^0: $P_i = p^0 Q_i$.

PAY-AS-BID AUCTION. Each bidder i assigned Q_i pays her winning bids: $P_i = \int_0^{Q_i} b_i(y)\,dy$.

Note that most other sealed-bid auction formats in the literature (most conspicuously, the multi-unit Vickrey auction) also satisfy the definition of a conventional auction.

Finally, the equilibrium concept used in this paper is the usual *Bayesian-Nash equilibrium*, which comprises a profile of bid functions, $b_i(q,t_i)$, for every type of every bidder which are mutual best responses. An equilibrium outcome, $Q_i(t)$, of a conventional auction is said to be *ex post efficient* if the good goes to the bidders with the highest values: $Q_i(t) \equiv Q_i^*(t)$ for all $i = 1, \ldots, n$, with probability one.

2 The Uniform-Price Auction is Inefficient

In this section, we develop the main result of the paper: all equilibria of the uniform-price auction are inefficient. We begin with the following lemma about all conventional auctions:

LEMMA 1. *Suppose, in a flat demands setting, that a Bayesian-Nash equilibrium of a conventional auction attains ex post efficiency. Then all bidders must use symmetric, flat bid functions: there exists a strictly increasing function* ϕ:[0,1]$\to$[0,1] *such that* $b_i(q,t_i) = \phi(t_i)$ *for all bidders* $i = 1,\ldots, n$ *, for all quantities* $q \in [0,\lambda_i]$, *and for almost every type* $t_i \in [0,1]$.

PROOF. First, we demonstrate that, in an ex post efficient equilibrium, each bidder must use a flat bid function, almost everywhere in types. Ex post efficiency requires that $Q_i = \lambda_i$ if bidder i has the highest type and $Q_i = 0$ if bidder i has the lowest type. Hence, for any $t_i' > t_i$, $b_i(\lambda_i,t_i') \geq b_i(0,t_i)$. Otherwise, when all other bidders' types, t_{-i}, lie strictly between t_i and t_i' (a positive-probability event, since the joint density of types t is strictly positive for $t \in (0,1)^n$), t_i' must win λ_i and t_i must win 0, but this cannot happen if $b_i(\lambda_i,t_i') < b_i(0,t_i)$. Defining $\phi_i(t_i) = ½\,[b_i(0,t_i) + b_i(\lambda_i,t_i)]$, this implies that $\phi_i(\cdot)$ is a weakly increasing function. Also define $T_i = \{t_i \in (0,1) \mid \phi_i(\cdot)$ is differentiable at $t_i\}$. Observe that $b_i(0,t_i) = b_i(\lambda_i,t_i)$ for every $t_i \in T_i$. Furthermore, since a monotonic function is differentiable almost everywhere, the measure of T_i equals one, for all $i = 1 , \ldots , n$. Since $b_i(\cdot,t_i)$ is weakly decreasing in q, we conclude that $b_i(\cdot,t_i)$ is constant in q for almost every type t_i.

Second, bidders use symmetric bid functions almost everywhere in types. Otherwise, there exist $i \neq j$ and $x \in T_i \cap T_j$ such that $\phi_i(x) < \phi_j(x)$. Using the continuity of $\phi_i(\cdot)$ and $\phi_j(\cdot)$ at x, there exist $t_i \in T_i$ and $t_j \in T_j$ such that $t_i > x > t_j$ but $\phi_i(t_i) < \phi_j(t_j)$. But then, when all other bidders' types, t_{-ij}, lie strictly between t_i and t_j (a positive-probability event), t_i must win λ_i and t_j must win 0, but this cannot happen if

$b_i(\cdot,t_i) = \phi_i(t_i) < \phi_j(t_j) = b_i(\cdot,t_j)$. We conclude that $\phi_i(x) = \phi_j(x)$ for almost every type x, and we write $\phi(x)$ for the common bid.

Third, $\phi(\cdot)$ is strictly increasing. Otherwise, there exist $x' > x$ such that $\phi(x') = \phi(x)$, and therefore t_i, $t_i' \in T_i$ such that $x' > t_i' > t_i > x$ and $\phi(t_i') = \phi(t_i)$. We can then repeat the same argument as above: when all other bidders' types, t_{-i}, lie strictly between t_i and t_i' (a positive-probability event), t_i' must win λ_i and t_i must win 0 for efficiency. But this cannot happen if $b_i(\cdot,t_i') = b_i(\cdot,t_i)$. ■

Our first theorem shows that, *in bids for sufficiently small quantities*, the uniform-price auction has the same intuition as the second-price auction for a single item: bidders optimally bid their true values. If bidder i's opponents all submit flat bids, then bidder i's bid is never "pivotal" in the uniform-price auction, in the sense that there exists L_i ($0 < L_i \le \lambda_i$) with the property that when bidder i wins a positive quantity, $b_i(q,t_i)$ never determines the market-clearing price for quantities in the subinterval $q \in [0,L_i]$. Analogous to the reasoning for the second-price auction of a single item, it will then be optimal for bidder i to bid $b_i(q,t_i) = w_i^+(t_i,t_i)$ of Definition 1. Furthermore, in light of Lemma 1, this bid function is independent of i and extends in an efficient equilibrium to the entire interval $q \in [0,\lambda_i]$. We have:

THEOREM 1. *Suppose, in a flat demands environment exhibiting the Generalized Winner's Curse, that there exists an ex post efficient equilibrium of the uniform-price auction. Then the expected value to bidder i conditional on winning a small quantity is independent of i, that is, there exists* $w^+(\cdot,\cdot)$ *such that* $w_i^+(x,x) = w^+(x,x)$, *for every bidder i* = 1, …, *n and every x* $\in [0,1]$. *Moreover, every bidder i uses the symmetric flat bid function* $b_i(q,t_i) = \phi(t_i) = w^+(t_i,t_i)$, *for every type* $t_i \in [0,1]$ *and every quantity* $q \in [0,\lambda_i]$.

PROOF. For each $i = 1, \ldots, n$, define:

$$J_{-i} = \underset{I\subset\{1,\ldots,n\}\setminus\{i\}}{\arg\max}\left\{\sum_{j\in I}\lambda_j \,\middle|\, \sum_{j\in I}\lambda_j < 1\right\} \text{ and } L_i = 1 - \sum_{j\in J_{-i}}\lambda_j . \tag{1}$$

(If there are multiple possible sets J_{-i}, select one arbitrarily.) With this, J_{-i} is a combination of bidders other than bidder i with a combined capacity, $1 - L_i$, closest to, but strictly less than, one (the total quantity available). Note that $L_i > 0$; if in eq. (1), $L_i = 1 - \sum_{j\in J_{-i}}\lambda_j > \lambda_i$, redefine $L_i = \lambda_i$.

By Lemma 1, in an efficient equilibrium, all bidders j submit flat bid schedules $b_j(q,t_j) = \phi(t_j)$ with probability one. Consider any bidder i, any quantity $q \in (0,L_i)$, and almost any $t_{-i} \in (0,1)^{n-1}$. Then by eq. (1), for any combination J of opponents of bidder i, we have $q + \sum_{j\in J}\lambda_j \ne 1$, establishing that bidder i's bid is not pivotal. Note, then, that $w_i^q(x,y)$ and $f_i^q(y \mid x)$ are constant in q on the interval $(0,L_i)$, for every

$x,y \in (0,1)$; consequently, $w_i^q(x,y) = w_i^+(x,y) \equiv \lim_{q \downarrow 0} w_i^q(x,y)$ and

$f_i^q(y \mid x) = f_i^+(y \mid x) \equiv \lim_{q \downarrow 0} f_i^q(y \mid x)$ for all $q \in (0,L_i)$.

Therefore,[6] bidder i's optimal strategy for $q \in [0,L_i)$ is to bid b which maximizes

$$L_i \int_0^{\phi^{-1}(b)} [w_i^+(x,y) - \phi(y)] f_i^+(y \mid x)\, dy. \tag{2}$$

Recall that $\phi(\cdot)$ of Lemma 1 is monotonic, and hence continuous almost everywhere. Consider any x at which $\phi(\cdot)$ is continuous, and also recall that for any such x, $b_i(q,x) = \phi(x)$, for all quantities $q \in [0,\lambda_i]$. Next, observe that the integrand of eq. (2) is independent of b and, in fact, b enters into the expression *only* through the upper limit, $\phi^{-1}(b)$, on the integral. Thus, if the bid $b = \phi(x)$ is optimal, it must be the case that the integrand, evaluated at $y = \phi^{-1}(b) = x$, equals zero. (Otherwise, since the integrand of eq. (2) is continuous in y when evaluated at $y = x$, there must exist $\varepsilon > 0$ such that either the integrand is positive for all $y \in (x,x+\varepsilon)$ or the integrand is negative for all $y \in (x-\varepsilon,x)$; either of these conclusions would contradict the optimality of $b = \phi(x)$.) But then, $\phi(x) = w_i^+(x,x)$. Moreover, this conclusion holds for every bidder, so that $w_i^+(t_i,t_i) = w_j^+(t_i,t_i) \equiv w^+(t_i,t_i)$, for all bidders $i, j = 1, \ldots, n$, and for almost every type $t_i \in [0,1]$. Finally, $\phi(t_i) = w_i^+(t_i,t_i)$ is strictly increasing in t_i,[7] so $b_i(q,t_i) = \phi(t_i) = w^+(t_i,t_i)$ for *every* type t_i. ■

If the auction environment exhibits interdependencies in values, the efficiency of the uniform-price auction requires a substantial amount of symmetry to the model: it is unlikely that $w_i^+(t_i,t_i)$ would be the same for all bidders unless the valuation functions, capacities and distributions of types were substantially symmetric.[8] Alternatively, if the auction environment exhibits pure private values, then Theorem 1 by

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[6] This is the main point in the proof where we assume the Generalized Winner's Curse. Conversely, suppose that $w_i^q(t_i,t_i) > w_i^+(t_i,t_i)$ for some $q \in (0,\lambda_i]$. Then it might be the case that bidder i would want to bid greater than $w_i^+(t_i,t_i)$ at some small $q \in (0,\lambda_i]$, in order that she be able to bid greater than $w_i^+(t_i,t_i)$ at some large $q \in (0,\lambda_i]$. (Recall that bids are constrained to be weakly decreasing in quantity.) However, the Generalized Winner's Curse excludes this as a possibility.

[7] This also assumes the Generalized Winner's Curse, for the condition that $w_i^q(x,y)$ is increasing in x and nondecreasing in y.

[8] At the same time, Theorem 1 does not mandate as much symmetry as assumed by Corollary 1. Consider a three-bidder auction with symmetric capacities, a symmetric type distribution, and valuations given by

$v_1(t) = 1/2\ t_1 + 1/3\ t_2 + 1/6\ t_3$;
$v_2(t) = 1/2\ t_2 + 1/3\ t_3 + 1/6\ t_1$;
$v_3(t) = 1/2\ t_3 + 1/3\ t_1 + 1/6\ t_2$.

Observe that each bidder's function $w_i^q(t_i,t_i)$ may now be identical—consistent with Theorem 1—but assumption (i) of Corollary 1 is not satisfied as, for example, $v_1(t_1,t_2,t_3) \neq v_1(t_1,t_3,t_2)$. Since the hypothesis of Corollary 1 generalizes the assumptions of Milgrom and Weber (1982), it should be observed that this example also does not satisfy Milgrom and Weber's assumptions.

itself does not require symmetric capacities and distributions for efficiency to be possible. Observe that Corollaries 1 and 2, below, treat both of these cases.

However, our next theorem shows that, *in bids for a sufficiently large quantity*, there is generally a positive probability that a bidder will simultaneously influence price and win a positive quantity. This provides an incentive for the bidder to bid below her marginal valuation on larger quantities, thereby upsetting efficiency. Figure 1 illustrates this demand reduction with flat demand curves, which is formalized in Theorem 2.

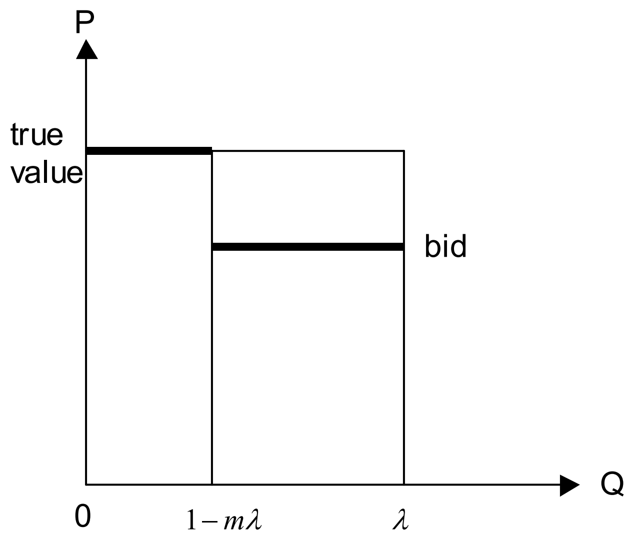

FIGURE 1. DEMAND REDUCTION IN THE UNIFORM-PRICE AUCTION

(CONSTANT MARGINAL VALUE FOR $q \in [0,\lambda]$)

THEOREM 2. *Consider a flat demands environment exhibiting the Generalized Winner's Curse. Unless* $\lambda_i = \lambda$ *for all* $i = 1, \ldots, n$ *and* $1/\lambda$ *is an integer, there does not exist an ex post efficient equilibrium of the uniform-price auction*.

PROOF. Suppose, to the contrary, that there exists an ex post efficient equilibrium of the uniform-price auction, but that either $\lambda_i = \lambda$, where $1/\lambda$ is not an integer, or $\lambda_1 > \lambda_n$. By Theorem 1, all bidders $i = 2, \ldots, n$ use the bid function $b_i(q,x) = \phi(x) = w^+(x,x)$, for all quantities $q \in [0,\lambda_i]$. Consider J_{-1} and L_1 defined in eq. (1). Although Theorem 1 showed that it is indeed a best response for bidder 1 to bid $b_1(q,x) = \phi(x) = w^+(x,x)$ for any $q \in [0,L_1]$, we now construct $R_1 \in (L_1,\lambda_1]$ with the property that it is *not* a best response for bidder 1 to bid $b_1(q,x) = \phi(x) = w^+(x,x)$, for any $q \in (R_1,\lambda_1]$, providing a contradiction.

Let us begin by observing that the interval $(L_1,\lambda_1]$ is nonempty. If $\lambda_i = \lambda$ for all i, this follows from a trivial calculation: let m be the greatest integer such that $m\lambda < 1$; then $L_1 = 1 - m\lambda$, which is strictly less than λ, since $1/\lambda$ is not an integer. If $\lambda_1 > \lambda_n$, define $j' = \max\{j \mid j \notin J_{-1}\}$. Observe that $j' \neq 1$, since $\lambda_2 + \ldots + \lambda_n \geq 1$. There are two cases: $\lambda_{j'} < \lambda_1$ (Case I); and $\lambda_{j'} = \lambda_1$ (Case II). In Case I, consider the set $J_{-1} \cup \{j'\}$. By the definition of L_1, we have $1 - L_1 + \lambda_{j'} \geq 1$. In Case II, observe that $j' \neq n$, so $n \in J_{-1}$. Consider the set $\{j'\} \cup J_{-1} \setminus \{n\}$. By the definition of L_1, we have $1 - L_1 + \lambda_{j'} - \lambda_n \geq 1$. In each case, this implies that $L_1 < \lambda_1$, as desired. Next, define

$$J' = \underset{I \subset \{2,\ldots,n\}}{\arg\min} \left\{ \sum_{j \in I} \lambda_j \;\middle|\; \sum_{j \in I} \lambda_j > 1 - \lambda_1 \right\} \text{ and } R_1 = 1 - \sum_{j \in J'} \lambda_j \,. \tag{3}$$

Since the set J_{-1} has the property that $\sum_{j \in J_{-1}} \lambda_j \equiv 1 - L_1 > 1 - \lambda_1$, observe that $1 - R_1 \leq 1 - L_1$, implying that $L_1 \leq R_1 < \lambda_1$. (For the case of $\lambda_i = \lambda$, where $1/\lambda$ is not an integer, $R_1 = L_1 = 1 - m\lambda$.) It is easy to see that $\tau_1^q(t_{-1})$, $F_1^q(y \mid x)$, $f_1^q(y \mid x)$ and $w_1^q(x,y)$ are constant in q for $q \in (R_1,\lambda_1]$, and we write $\tau_1^2(t_{-1})$, $F_1^2(y \mid x)$, $f_1^2(y \mid x)$ and $w_1^2(x,y)$, respectively, for these values. There also exists an $\varepsilon > 0$ such that $\tau_1^q(t_{-1})$, $F_1^q(y \mid x)$, $f_1^q(y \mid x)$ and $w_1^q(x,y)$ are constant in q for $q \in (R_1-\varepsilon, R_1)$, and we write $\tau_1^1(t_{-1})$, $F_1^1(y \mid x)$, $f_1^1(y \mid x)$ and $w_1^1(x,y)$, respectively, for these values. In the notation of Definition 1, $w_1^+(x,x) = w_1^1(x,x)$.

Given type $x \in (0,1]$, consider an alternative strategy for bidder 1 of bidding $b = \phi(x)$ for $q \in [0,R_1]$, and bidding $\beta \leq \phi(x)$, for $q \in (R_1,\lambda_1]$. Let $\Pi_1(\beta)$ denote the expected payoff from this two-step strategy given that the other firms are bidding $b_i(q,\cdot) = \phi(\cdot)$. We show that $d\Pi_1(\beta)/d\beta$ evaluated at $\beta = \phi(x)$ is negative, which implies that bidder 1 can strictly improve her payoff by using the two-step bid function with $\beta < \phi(x)$, rather than bidding $\phi(x)$ for all $q \in [0,\lambda_1]$. There are three regions of bidder types to consider in calculating $\Pi_1(\beta)$:

1. $\phi^{-1}(\beta)$ is greater than $\tau_1^2(t_{-1})$. Then bidder 1 wins quantity λ_1, and $\tau_1^2(t_{-1})$ determines the price. The contribution to the expected payoff is:

$$\lambda_1 \int_0^{\phi^{-1}(\beta)} [w_1^2(x,y) - w_1^+(y,y)]\, f_1^2(y|\, x)\, dy \,.$$

2. $\phi^{-1}(\beta)$ is between $\tau_1^1(t_{-1})$ and $\tau_1^2(t_{-1})$. Then bidder 1 wins quantity R_1, and β determines the price. The contribution to the expected payoff is

$$R_1 \int_0^{\phi^{-1}(\beta)} \int_{\phi^{-1}(\beta)}^1 [E(V_1 \,|\, t_1 = x, \tau_1^2(t_{-1}) = y, \tau_1^1(t_{-1}) = z) - \beta]\, f_1^{2,1}(y,z|\, x)\, dy\, dz \,,$$

where $f_1^{2,1}(y,z \mid x)$ denotes the joint density of $\tau_1^2(t_{-1}) = y$ and $\tau_1^1(t_{-1}) = z$, conditional on $t_1 = x$.

3. $\phi^{-1}(\beta)$ is less than $\tau_1^{\,1}(t_{-1})$. Then when x is greater than $\tau_1^{\,1}(t_{-1})$, bidder 1 wins quantity R_1, and $\tau_1^{\,1}(t_{-1})$ determines the price. The contribution to the expected payoff is

$$R_1 \int_{\phi^{-1}(\beta)}^{x} (w_1^{\,1}(x,z) - w_1^{\,+}(z,z))\, f_1^{1}(z\,|\,x)\, dz\ .$$

$\Pi_1(\beta)$ is the sum of these three integrals. Taking the derivative of each with respect to β when evaluated at $\beta = \phi(x)$, and combining and simplifying terms yields:

$$\frac{d\Pi_1}{d\beta} \;=\; -R_1\ \Pr\{\tau_1^{\,1}(t_{-1}) < x < \tau_1^{\,2}(t_{-1})\} + (\lambda_1 - R_1)\, \phi^{-1\prime}(\phi(x))\,[w_1^{\,2}(x,x) - w_1^{\,+}(x,x)]\, f_1^{2}(x|\,x)\ . \qquad (4)$$

Observe that the first term of the right-hand-side of eq. (4) is strictly negative, while the second term is weakly negative (since $w_1^{\,2}(x,x) \le w_1^{\,+}(x,x)$, from the Generalized Winner's Curse assumption). Hence, bidder 1 strictly gains by bidding $\beta < \phi(t_1)$ for $q \in (R_1,\lambda_1]$, so it cannot be an equilibrium for each bidder to bid the symmetric flat bid function $\phi(t_1)$. But by Theorem 1, bidding $\phi(t_1)$ is the only candidate for an ex post efficient equilibrium, yielding a contradiction. ■

The intuition behind Theorem 2 is that bidders have market power in the uniform-price auction. If a bidder has a positive probability of influencing price in a situation where the bidder wins a positive quantity, then the bidder has an incentive to shade her bid. This intuition is formalized in the proof. Bidder i cannot be pivotal for quantities $q \in [0,L_i]$, so bidder i bids her expected value on the left subinterval. However, bidder i is pivotal with positive probability for quantities $q \in (R_i,\lambda_i]$, so bidder i gains by shading her bid on the right subinterval. Her marginal gain from shading consists of two terms: (1) the quantity at which the bidder becomes pivotal times the probability that her bidder is pivotal; and (2) the Generalized Winner's Curse effect.

If $1/\lambda$ is an integer and $\lambda_i = \lambda$, then the proof of inefficiency does not go through. In this special case, $1 - m\lambda$ of Figure 1 is no longer an interior point. Since the bidder only affects price when she wins nothing, bidding her value is a best response and efficiency is achieved. However, this is a knife-edge result. Efficiency is the exception, not the rule. Efficiency is lost in essentially all settings outside this special case, which is the same as the unit demand model.

COROLLARY 1 (SYMMETRIC INTERDEPENDENT VALUES MODEL). *Consider any flat demands setting that additionally satisfies:*

(i) *Symmetric value functions: if* $\pi_1, \ldots, \pi_n$ *is a permutation of* $1,\ldots,n$*, then* $v_{\pi_1}(t_{\pi_1}, \ldots, t_{\pi_n}) = v_1(t_1, \ldots, t_n)$*;*

(ii) *Symmetric joint distribution: if* $\pi_1, \ldots, \pi_n$ *is a permutation of* $1,\ldots,n$, *then* $F(t_{\pi_1}, \ldots, t_{\pi_n}) = F(t_1, \ldots, t_n)$ *;*

(iii) *Affiliated types:* $t_1,\ldots, t_n$ *are affiliated random variables; and*

(iv) *Symmetric capacities:* $\lambda_1 = \cdots = \lambda_n \equiv \lambda$.

Unless $1/\lambda$ *is an integer, there does not exist an ex post efficient equilibrium of the uniform-price auction.*

PROOF. From (i), each bidder's value may be written as the same function, $v(\cdot)$, of the bidder's own type and the order statistics of the other bidders' types. From (ii) and (iii), the order statistics of the other bidders' types are affiliated random variables (Milgrom and Weber 1982, Theorem 2). From (iv), $\tau_i^1(t_{-i})$ and $\tau_i^2(t_{-i})$ from the proof of Theorem 2 simply equal $t^{-i}_{(m+1)}$ and $t^{-i}_{(m)}$, respectively, the $(m+1)^{\text{st}}$ and m^{th} order statistics of the other bidders' types. Since the expectation of a (weakly) increasing function of affiliated random variables is monotonic in each component (Milgrom and Weber 1982, Theorem 5), it easily follows that $E[V_i \mid t_i = x, t^{-i}_{(m)} = x] \le E[V_i \mid t_i = x, t^{-i}_{(m+1)} = x]$; that is, the Generalized Winner's Curse holds, and Theorem 2 applies. ■

COROLLARY 2 (PRIVATE VALUES MODEL). *Consider a flat demands environment that additionally satisfies:* $v_i(t_i,t_{-i}) = t_i$, *for each* $i = 1,\ldots, n$. *Unless* $1/\lambda$ *is an integer and* $\lambda_i = \lambda$ *for all i, there does not exist an ex post efficient equilibrium of the uniform-price auction.*

PROOF. The Private Values Model exhibits the Generalized Winner's Curse, since $E[V_i]$ is independent of t_{-i} and, so, is constant in the quantity won by bidder i. Again, Theorem 2 applies.■

Furthermore, observe that the Pure Private Values Model does not actually require the "flat demands" assumption for the inefficiency theorem to hold (e.g., see Theorem 3 below and Bolle (1997) for treatments of independent private values and downward-sloping demands).

3 The Ranking of Conventional Auctions is Ambiguous

It is often been claimed—especially in discussions of the U.S. Treasury auctions—that the uniform-price auction is superior to the pay-as-bid auction for selling multiple items (Back and Zender 1993 is an exception). We have seen above that this intuition, which derives largely from the analysis of auctions where bidders have tastes for only a single item, is flawed. In uniform-price auctions, rational bidders will bid strategically by submitting lower unit prices for larger quantities than for smaller quantities, even in contexts where demands are flat.

In this section, we go a step further by establishing that in some reasonable situations where efficiency is impossible in a uniform-price auction, full efficiency is nevertheless possible in a pay-as-bid auction. The intuition is straightforward. Our inefficiency result is driven by the incentive for demand reduction in the uniform-price auction: a bidder who shades her bids on subsequent units saves money on the purchase of earlier units. By contrast, this incentive does not exist in the pay-as-bid auction: a bidder who reduces her demand for subsequent units (but holds her bids constant on earlier units) does not realize any savings on her purchase of earlier units. A bidder's demand reduction may reduce the market-clearing price, but this provides no help to her other purchases, which are made at the prices she bid.

This is analogous to the situation of a monopolist deciding how much to produce. Recall that the uniform-price auction is often referred to as a "nondiscriminatory auction" while the pay-as-bid auction is often referred to as a "discriminatory auction." Just as monopoly without price discrimination leads to social inefficiency but a monopoly with perfect price discrimination may realize all gains from trade, a nondiscriminatory auction will lead to inefficiency but a discriminatory auction has the possibility of efficiency. The nondiscriminating monopolist's marginal revenue curve lies strictly below her demand curve, except at a zero quantity; whereas the perfectly-discriminating monopolist's marginal revenue curve may actually coincide with her demand curve. So we obtain supply reduction in the former situation, but not necessarily in the latter situation.

To simplify the analysis, we assume for the development of Theorem 3 that bidders have flat demands, independent private values, and are *ex ante* symmetric. Thus, their marginal valuations v_i are i.i.d., and their capacities λ_i are equal. If bidder i obtains q units of the good for a total payment of P, she obtains payoff $u_i(v_i,q,P) = qv_i - P$, where $q \in [0,\lambda]$ and $\lambda \in (0,1)$. The distribution function F is commonly known, but the realization v_i is known only to bidder i.

We construct an efficient Bayesian-Nash equilibrium of the pay-as-bid auction using the following simple logic. Let $U_i(v_i)$ denote the interim expected utility of bidder i, and let $Q_i(v_i)$ denote the interim expected quantity received by bidder i, in an allocatively-efficient direct mechanism. As before, define m to be the greatest integer less than $1/\lambda$, let $v_{(m)}^{-i}$ denote the m^{th} order statistic of all bidders except i, and let $F_{(m)}^{-i}(\cdot)$ denote its distribution function. Observe that efficiency requires that bidder i must obtain λ units of the good if $v_i > v_{(m)}^{-i}$, $1 - m\lambda$ units of the good if $v_{(m+1)}^{-i} < v_i < v_{(m)}^{-i}$, and 0 units of the good if $v_i < v_{(m+1)}^{-i}$. Thus,

$$Q_i(v_i) = \lambda F_{(m)}^{-i}(v_i) + (1 - m\lambda)\left[F_{(m+1)}^{-i}(v_i) - F_{(m)}^{-i}(v_i)\right].$$

Since the interim expected utility of the zero type must equal zero, the usual incentive-compatibility argument implies that $U_i(v_i)$ is given by

$$U_i(v_i) = \int_0^{v_i} Q_i(x)\,dx\,. \tag{5}$$

Now suppose that an efficient equilibrium of the pay-as-bid auction exists. By Lemma 1, each bidder must use a flat bid function almost everywhere: $b_i(q,v) = \phi_i(v)$. Using this bid function, a second way to calculate the interim expected utility of bidder i is

$$U_i(v_i) = Q_i(v_i)[v_i - \phi_i(v_i)]\,. \tag{6}$$

Combining eqs. (5) and (6) gives us

$$\phi_i(v_i) = v_i - \frac{\int_0^{v_i} Q_i(x)\,dx}{Q_i(v_i)}\,. \tag{7}$$

We thus have

THEOREM 3. *If bidders' valuations are i.i.d. and their capacities*, λ_i, *are equal, then eq*. (7) *provides an ex post efficient equilibrium of the pay-as-bid auction*.

PROOF. The above argument showed that a necessary condition for an ex post efficient equilibrium of the pay-as-bid auction is eq. (7). If v_i and v_j are i.i.d. and $\lambda_i = \lambda_j$, then $Q_i(\cdot) = Q_j(\cdot)$ and $\phi_i(\cdot) = \phi_j(\cdot) \equiv \phi(\cdot)$. Furthermore, $\phi(\cdot)$ is strictly monotone increasing, so every bidder using the same bid function, $\phi(\cdot)$, leads to an efficient allocation. Finally, it is easily verified that every bidder using $\phi(\cdot)$ constitutes a Bayesian-Nash equilibrium. ■

However, the positive result of Theorem 3, below, does not mean that pay-as-bid pricing should be preferred to uniform pricing. Consider the oft-studied problem of the first-price auction for a single indivisible item. It is well known that very special conditions are necessary for the first-price auction to admit an efficient equilibrium. Instead, if bidders' valuations are random variables that are not identically distributed, then any equilibrium of the first-price auction will typically be inefficient. These considerations from the first-price auction equally carry over to the current context. Thus, the assumption in Theorem 3 that each bidder's marginal valuation, v_i, is drawn from the *same* distribution should be viewed as essential. Theorem 4 treats the case of asymmetric bidders, and easily obtains a negative result. All we argue here is merely that—counter to the conventional wisdom—the ranking between pay-as-bid and uniform pricing is ambiguous, both on efficiency and revenue grounds.

THEOREM 4. *If bidders' valuations are independent but not identically distributed, or if their capacities are unequal, then generically there does not exist an ex post efficient equilibrium of the pay-as-bid auction*.

PROOF. Suppose there exist bidders i and j such that the associated distribution functions, $F_i(\cdot)$ and $F_j(\cdot)$, are not identical. As before, a necessary condition for an ex post efficient equilibrium is that bidder i's bid function be given by $\phi_i(\cdot)$, defined by the right-hand-side of (7). At the same time, another necessary condition is that bidder j's bid function be given by $\phi_j(\cdot)$, defined by replacing $F_{(m)}^{-i}$ and $F_{(m+1)}^{-i}$ with $F_{(m)}^{-j}$ and $F_{(m+1)}^{-j}$ in the right-hand-side of (7). For generic $F_j \neq F_i$, the implied $\phi_j \neq \phi_i$ on sets of positive measure, contrary to Lemma 1. We conclude that there cannot exist any ex post efficient equilibrium.

Similarly, if the capacities λ_i are not all equal, then eq. (7) again implies that, if $\lambda_j \neq \lambda_i$, then $\phi_j \neq \phi_i$ on sets of positive measure. Hence, there again cannot exist an ex post efficient equilibrium. ■

Our discussion in this section has emphasized the goal of allocative efficiency. However, the ambiguous ranking of the uniform-price and pay-as-bid auctions equally holds if the objective is to maximize the seller's revenue. With symmetric bidders and flat demands, the revenue-maximizing auction—subject to the constraint that the seller dispose of all quantity—awards all quantity to the buyers who value them the most (Ausubel and Cramton 1999, Proposition 1). Furthermore, all inefficient assignments—subject to the same constraint—yield strictly lower seller revenue. Thus, revenue maximization coincides with efficiency in this environment. We have:

COROLLARY 3. *The efficiency and revenue rankings of the uniform-price and pay-as-bid auctions are inherently ambiguous.*

PROOF. Let $\lambda_i = \lambda$ for all i, $1/\lambda$ *not* an integer, and let the distributions be symmetric. By Theorems 2 and 3, the pay-as-bid auction has an equilibrium which outperforms all equilibria of the uniform-price auction on efficiency and, hence, revenues. However, let $1/\lambda$ be an integer, but let the distributions be asymmetric. Then the uniform-price auction has an efficient equilibrium which, by Theorem 4, outperforms all equilibria of the pay-as-bid auction on efficiency and, hence, revenues. ■

4 Inefficiency with Downward-Sloping Demands

In the preceding sections, we considered uniform-price auctions where bidder i's marginal values were constant over quantities $q_i \in [0,\lambda_i]$, but then discontinuously dropped to zero for quantities $q_i \in (\lambda_i,1]$. In this section, we reconsider the inefficiency argument in a pure private values model where bidders' marginal values are smoothly decreasing in quantity. We find that the uniform-price auction is still inefficient, and that essentially the same intuition as in the flat demands case applies.

Let t_i denote the type of bidder i. We assume the types $\{t_1,\ldots,t_n\}$ are drawn independently according to the distribution functions $\{F_1,\ldots,F_n\}$, where each F_i has positive and finite density f_i on support [0,1]. Each distribution function F_i is commonly known, but the realization t_i is known only to bidder i. If type t_i of bidder i obtains $q \in [0,1]$ units of the good for a total payment of P, her payoff is given by $V_i(q,t_i) - P$. The valuation functions $V_i(\cdot,\cdot)$ are required to have continuous partial derivatives with respect to q, which we denote by $v_i(\cdot,\cdot)$. For every bidder $i = 1, \ldots, n$, we assume the marginal value functions $v_i(\cdot,\cdot)$ satisfy:

(a) $v_i(\cdot,\cdot)$ is continuous in its two arguments and has continuous partial derivatives in each;

(b) $v_i(0,t_i) = t_i$, for all $t_i \in [0,1]$;

(c) $v_i(q,0) = 0$, for all $q \in [0,1]$;

(d) $\partial v_i(q,t_i)/\partial t_i \geq 0$, for all $t_i \in [0,1]$ and for all $q \in [0,1]$;

(e) $\partial v_i(q,t_i)/\partial q \leq 0$, for all $t_i \in [0,1]$ and for all $q \in [0,1]$.

In addition, we require that at least one bidder $j \in \{1,\ldots,n\}$ satisfy:

(d′) $\partial v_j(q,t_j)/\partial t_j > 0$, for all $t_j \in [0,1]$ and for all $q \in [0,1]$;

(e′) $\partial v_j(q,t_j)/\partial q < 0$, for all $t_j \in (0,1]$ and for all $q \in [0,1]$.

Thus, marginal values are weakly increasing (strictly for at least one bidder) in own type, and type has the literal interpretation of the bidder's marginal value at zero units. Furthermore, each type exhibits weakly decreasing (strictly for nonzero types of at least one bidder) marginal value in quantity.

Our approach shall be to suppose that there exists an equilibrium in the uniform-price auction that attains efficiency, and to demonstrate that this leads to a contradiction. Taking a mechanism-design-like approach, let us suppose that a mediator asks every bidder $j \neq i$: "What quantity, $q_j(v,t_j)$, gives you a marginal value of v?" Let us suppose further that all bidders $j \neq i$ respond truthfully to this question. Finally, the mediator uses these responses together with the response of bidder i (who is not required to be truthful) to attempt to allocate all the available quantity ($S = 1$) efficiently.

Let $G(v;y)$ denote the probability that, under truthtelling by all bidders $j \neq i$, and under efficient allocation by the mediator, a request by bidder i for quantity y at a marginal value of v is filled. (The identity "i" of bidder i is really a part of $G(v;y)$, but it is suppressed from eq. (8) *et seq* for notational brevity.) To be precise,

$$G(v;y) = \Pr\{\sum_{j\neq i} q_j(v,t_j) \leq 1 - y\}\,. \tag{8}$$

Note that the function, $G(v;y)$, is determined solely by the bidders' marginal values and the distribution of bidders' types. It asks what is the probability of a particular request (v,y) being filled if the other bidders report truthfully and if the mediator carries out the efficient allocation conditional on the reports.

Consider a uniform-price auction, and let $b_i(q,t_i)$ denote the bid (the price per unit) submitted for quantity q by type t_i of bidder i. As in Section 1, we require the bids $b_i(\cdot,t_i)$:[0,1]→[0,1] to be right continuous at $q = 0$, left continuous at all $q \in (0,1]$, and weakly decreasing functions of q. Analogous to Lemma 1, we have

LEMMA 2. *In any ex post efficient equilibrium of the uniform-price auction, bids are a strictly monotone function of marginal value whenever* $G(v;y) \in (0,1)$; that is,

$$b_i(q,t_i) = \phi(v_i(q,t_i)), \text{ where } \phi:[0,1]\to[0,1] \text{ is strictly monotone.} \tag{9}$$

The intuition for Lemma 2 is that, in a uniform-price auction (or any conventional auction), quantity is allocated to the highest bids. In order for the auction to place the goods in the hands who value them most, bids for any given quantity must be monotone in the marginal value. The meaning of $G(v,y) \in (0,1)$ is simply that this logic applies for all bids that are not sure to win or sure to lose—changes in rankings of such bids have no effect on either the allocation or the payment in a uniform-price auction, so monotonicity in marginal value is unnecessary if there is no possibility the bid is pivotal.

Let us assume that $\phi(\cdot)$ defined by eq. (9) is continuously differentiable, and let us derive the Euler equation expressing the requirement that bidder i is optimizing in her quantity, $y(v)$, requested for each v. The problem for type t_i of bidder i is to determine the function $y(v)$ that maximizes her expected payoff:

$$\max_{y(\cdot)} \int_0^\infty \{V_i(y,t_i) - \phi(v)y\}\, dG(v;y)\,. \tag{10}$$

Integrating by parts, the optimization problem (10) may be rewritten as

$$\max_{y(\cdot)} \int_0^\infty \{\phi'(v)y - [v_i(y,t_i) - \phi(v)]y'\} G(v;y)\, dv\,. \tag{11}$$

Thinking of the integrand in (11) as a function of v, $y(v)$, and $y'(v)$, we can characterize the solution to this maximization problem by the Euler equation

$$[v_i(y,t_i) - \phi(v)]G_v(v;y) + y\phi'(v)G_y(v;y) = 0\,, \tag{12}$$

where $G_v(v;y)$ and $G_y(v;y)$ denote the partial derivatives of $G(v;y)$, defined in (8), with respect to v and y.

We still need to impose the additional requirement that bidder i report truthfully. For the outcome to be efficient, it must be the case that truthtelling satisfies the Euler equation (12) for bidder i. Substituting $q_i(v,t_i)$ in place of y, in eq. (12) yields

$$[v-\phi(v)]G_v(v;q_i(v,t_i))+q_i(v,t_i)\phi'(v)G_y(v;q_i(v,t_i))=0\,. \tag{13}$$

LEMMA 3. *Assume that every bidder* $i = 1, \ldots, n$ *satisfies conditions* (a)–(e), *and at least one bidder* $j \neq i$, *also satisfies* (d′)–(e′). *Then for every* (v,y) *such that* $G(v;y) \in (0,1)$:

$$G_v(v;y)>0\text{, and} \tag{14}$$

$$G_y(v;y)<0\,. \tag{15}$$

PROOF. Without loss of generality, let $j = 2$ satisfy (d′)–(e′) and define $G(v;y)$ with respect to $i = 1$. First, consider the case with two bidders. Then $G(v;y) = F_j[t_j(v,1-y)]$, where $t_j(v,z)$ is defined implicitly by $v_j(z,t_j(v,z)) = v$; that is, $t_j(v,z)$ denotes the type who has marginal value v when consuming z units. By the implicit function theorem, and using (d′)-(e′), $\partial t_j(v,z)/\partial v > 0$ and $\partial t_j(v,z)/\partial z > 0$ whenever $G(v,z) \in (0,1)$. Thus, using the fact that the density is positive and bounded everywhere in the support, $G_v(v;y) = f_j[t_j(v,1-y)]\, \partial t_j(v,1-y)/\partial v > 0$ and $G_y(v;y) = -f_j[t_j(v,1-y)]\partial t_j(v,1-y)/\partial(1-y) < 0$.

Second, we establish the inductive step: if $G_v(v;y) > 0$ and $G_y(v;y) < 0$ when there are $m - 1$ bidders, then the same inequalities hold when there are m bidders. Define

$$G^{m-1}(v;y)=\Pr\{\sum_{k=2}^{m-1}q_k(v,t_k)\le 1-y\}\text{ and }G^{m}(v;y)=\Pr\{\sum_{k=2}^{m}q_k(v,t_k)\le 1-y\}\,.$$

Observe that

$$G^m(v;y)=\int_0^{t_m(v,1-y)}G^{m-1}(v;y+q_m(v,t))f_m(t)\,dt\,. \tag{16}$$

Differentiating (16) with respect to each of v and y, and using the inductive hypothesis that $G_v^{m-1}(v;y) > 0$ and $G_y^{m-1}(v;y) < 0$, allows us to conclude that $G_v^m(v;y) > 0$ and $G_y^m(v;y) < 0$. By induction, we conclude that inequalities (14) and (15) hold when $G^n(v;y)$ is calculated using all n bidders. ■

THEOREM 5. *Assume that every bidder* $i = 1, \ldots, n$ *satisfies conditions* (a)–(e) *and at least one bidder* $j \in \{1,\ldots,n\}$ *also satisfies* (d′)–(e′). *Then there does not exist an ex post efficient equilibrium of the uniform-price auction*.

PROOF. Suppose to the contrary that an efficient equilibrium exists. Select any $i \neq j$, and define $G(v;y)$ with respect to bidder i. Consider any $v^0 \in (0,1)$ with the property that $G(v^0;0) < 1$. Such a v^0 clearly exists under (d′)–(e′), and each $v \in (0,v^0)$ has this same property. Observe that Lemma 3 implies that $G_v(v;0) > 0$ and $G_y(v;0) < 0$ whenever $v \in (0,v^0)$. We showed above that a necessary condition for an efficient equilibrium is the Euler equation (13). For each $v \in (0,v^0)$, we can substitute $(v,t_i) = (v,v)$ into (13). Noting that $q_i(v,v) = 0$, we conclude that $\phi(v) = v$ for all $v \in (0,v^0)$.

However, we may instead substitute any $v \in (0,v^0)$ paired with any $t_i \in (v,1]$ into (13). Since $q_i(v,t_i) > 0$, $\phi'(v) = 1$, and $G_y(v;q_i(v,t_i)) < 0$, the second term of (13) is strictly negative. In order to offset this, the first term of (13) would need to be strictly positive. Given that $G_v(v;q_i(v,t_i)) > 0$, this would require that $v - \phi(v) > 0$. But we have already concluded that $\phi(v) = v$, yielding a contradiction to our hypothesis that there existed an efficient equilibrium. ■

The intuition behind Theorem 5 is straightforward. At quantities of zero, bidders in an efficient equilibrium would optimally bid their true marginal values: $b_i(0,t_i) = v_i(0,t_i) = t_i$, as in the second-price auction for a single item. However, at strictly positive quantities $q > 0$, bidders in an efficient equilibrium would optimally shade their bids: $b_i(q,t_i) < v_i(q,t_i)$, and the amount of shading is increasing in q. The reason for this "differential shading" is that the incentive to win units at any price below marginal value is offset by the incentive to reduce the price paid on (infra-marginal) units that are going to be won anyway. At zero quantity, there are no infra-marginal units; but as q increases, the incentive to reduce the price on infra-marginal units becomes increasingly important. However, an efficient equilibrium also requires that there be a consistent mapping from marginal values to bids, or else there is no way for the auction to assign the goods to the bidders with the highest marginal values. Thus, we reach a contradiction to efficiency.

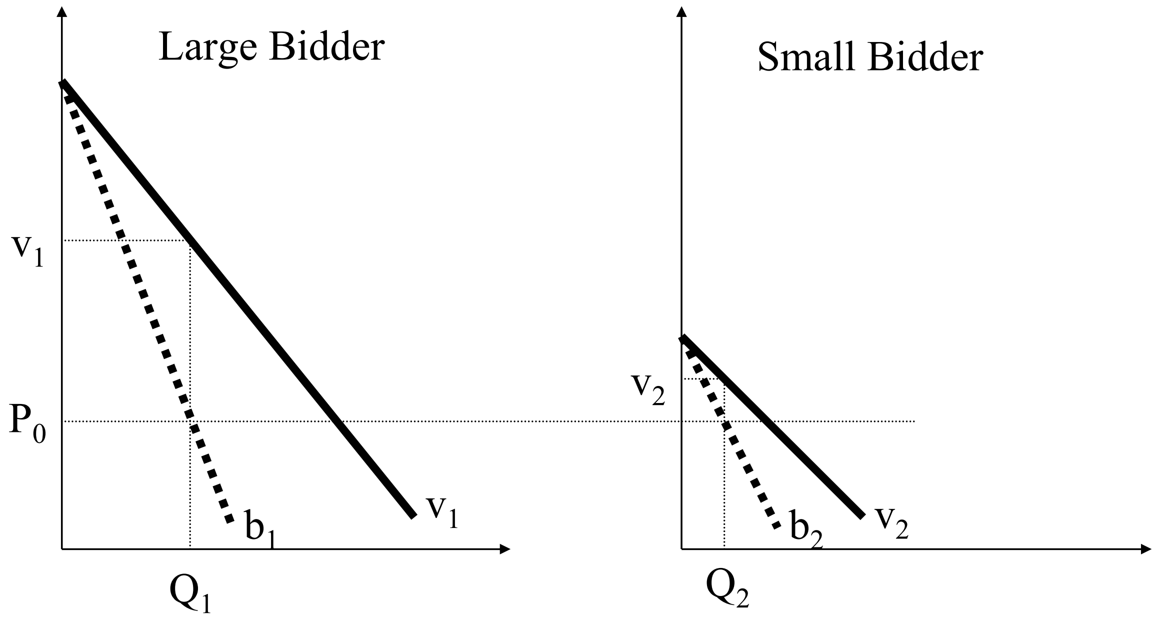

FIGURE 2. DEMAND REDUCTION IN THE UNIFORM-PRICE AUCTION
(DIMINISHING MARGINAL VALUES)

Figure 2 illustrates demand reduction and the associated inefficiency resulting from differential shading. The Euler equation (13) requires that the bid curves intersect the marginal-value curves at $q = 0$, but requires that the bid curves lie strictly below the marginal-value curves at all positive quantities. Moreover, as argued in the previous paragraph, the incentive to shade increases in quantity. Thus, the large bidder is shading more than the small bidder at the clearing price P_0. Since quantity is assigned based on the bids, not on the values, the large bidder wins too little and the small bidder wins too much. Indeed, efficiency in the example of Figure 2 would dictate that the large bidder win the entire quantity $Q_1 + Q_2$, but with uniform pricing, the large bidder has an incentive to make room for her smaller rival.

5 Examples

We have said little about existence and uniqueness of equilibria, nor about how one might go about constructing equilibria. In general, all of these remaining tasks may be problematical. In particular, we know from Wilson (1979) and subsequent papers that when the items are infinitely divisible, a vast multiplicity of equilibria is probably inherent to the uniform-price auction and, in any case, the calculation of equilibria may be difficult.

In this section, we give some examples where the above difficulties disappear. These examples—which have simple theoretical outcomes—are especially useful for experiments and classroom exercises.

The examples also are useful in getting a sense of the theoretical magnitudes of the necessary efficiency losses—and possible revenue losses—inherent in the uniform-price auction.

EXAMPLE 1: ONLY ONE MULTI-UNIT BIDDER. Consider a uniform-price auction with n+1 bidders and m *indivisible* identical items, where $n \geq m$. Bidders 1, …, n possess positive marginal values only for a single unit, but Bidder 0 possesses positive marginal values for two or more units. This is a simple extension of the unit-demand model (Weber 1983). As in that model, a single round of elimination of weakly dominated strategies has substantial cutting power. Each bidder 1, …, n finds that her bid cannot be pivotal in any state of the world in which she wins a unit, and so she bids her true value. After this round of elimination, Bidder 0 faces a simple decision problem whose solution, by Theorem 2, inevitably involves demand reduction.

Further suppose bidder 0 places the same marginal value, v_0, on each unit, where v_0 is drawn from distribution function F_0. Each bidder $i = 1,\dots,n$ demands only one unit of the good, placing a marginal value of v_i on the single unit, where each v_i is independently drawn from distribution function F. The distribution functions F_0 and F have the same support.

Let $v_{(k)}$ denote the k^{th}-order statistic of $v_1,\dots, v_n$ (the k^{th}-highest value excluding bidder 0). Bidder 0 performs a calculation analogous to that in the proof of Theorem 2. On her first unit of quantity, bidder 0 bids v_0. Let b denote her bid on the second unit, and let $\pi_0(v_0,b)$ denote her expected payoff from bidding b when her true value is v_0. Then

$$\pi_0(v_0,b) = 2\int_0^b (v_0 - p)\,dF_{(m-1)}(p) + [v_0 - b][F_{(m)}(b) - F_{(m-1)}(b)] + \int_b^{v_0} (v_0 - p)\,dF_{(m)}(p)\,.$$

Differentiating with respect to b and canceling terms yields the first-order condition

$$[v_0 - b]f_{(m-1)}(b) = F_{(m)}(b) - F_{(m-1)}(b)\,.$$

Recognizing that $F_{(m)}(b) - F_{(m-1)}(b) = \binom{n}{m-1}[1-F(b)]^{m-1}[F(b)]^{n-m+1}$, we conclude

$$b + \left(\frac{1}{m-1}\right)\left(\frac{1-F(b)}{f(b)}\right) = v_0\,. \tag{17}$$

If eq. (17) yields a unique b for each realization v_0, then the model in which Bidder 0 possesses positive marginal values for two units has a unique equilibrium in weakly-undominated strategies. When the implied b is nonnegative, this gives Bidder 0’s bid; when the implied b is negative, Bidder 0 bids zero.

EXAMPLE 2: ONLY ONE MULTI-UNIT BIDDER AND UNIFORM DISTRIBUTIONS. Continue Example 1, assuming $F(b) = b$. Eq. (17) then yields

$$b(v_0) = \begin{cases} \dfrac{(m-1)v_0 - 1}{m-2}, & \text{if } v_0 > \frac{1}{m-1} \\ 0\ , & \text{otherwise.} \end{cases} \tag{18}$$

EXAMPLE 3: ALL BIDDERS DESIRE TWO UNITS, SUPPLY EQUALS TWO UNITS, AND UNIFORM DISTRIBUTIONS. Continue Example 2 by assuming that the supply equals two units ($m = 2$). Observe that eq. (18) implies that if one multi-unit bidder, who has positive marginal value for two units, bids against n bidders, who each demand only a single unit, then $b(v_0) \equiv 0$. Thus, the two-unit bidder behaves in equilibrium as if he has a positive marginal value for only a single unit, and he bids his true value on the single unit.

Now suppose instead that each of the bidders desires two units. If all other bidders bid their true value on the first unit but zero on the second unit, the best response for the remaining bidder is also to bid her true value on the first unit but zero on the second unit. Thus, one equilibrium of the uniform-price auction is for every bidder to bid her true value on the first unit but to bid $b = 0$ on the second unit. (Equilibria with this structure were discovered by Noussair (1995) and Engelbrecht-Wiggans and Kahn (1998), in models closely related to Example 3.)

Example 3 is particularly striking in that it admits another equilibrium. Observe that it can be reinterpreted as a flat-demands model with $\lambda = 1$ for all bidders. Consequently, it is also an equilibrium for every bidder to bid her true value on the first unit and to also bid her true value on the second unit. That is to say, Example 3 lies within one of the exceptions to the Inefficiency Theorem, but it nevertheless possesses a grossly inefficient equilibrium as well as a fully efficient equilibrium.

6 Conclusion

Multi-unit auctions differ from single-unit auctions in essential ways. Most fundamentally, the efficiency result for the second-price auction of a single item does not extend to the uniform-price auction of many items. In a uniform-price auction, winning bidders affect the market price with positive probability. Hence, bidders have incentives to reduce their demands, upsetting both the strategic simplicity and the efficiency of uniform-price auctions. By shading her bid for marginal items, the bidder is able to reduce the expected price paid on inframarginal items. The more one buys, the greater the incentive to shade. As a result, large bidders will sometimes lose against small bidders on items that the large bidders value higher.

This intuition is easily understood. Indeed, it is similar to the basic result of monopoly—that a monopolist's marginal revenue curve lies below the demand curve. Yet, as we have seen in the Introduction, even Nobel laureates succumb to the fallacy that bidders have no incentive to shade their bids in the uniform-price auction.

In this paper, we prove the general inefficiency of the uniform-price auction. Differential incentives to shade bids arise whenever a winner influences the market-clearing price with positive probability. The only cases that escape our inefficiency result are: (1) pure common values, in which *all* assignments are efficient, and (2) single-unit demands and analogous cases, where a bidder determines the price only when the bidder wins zero quantity. Although these are the cases often studied in the multi-unit auction literature, their study stems more from tractability than from practical importance. In practice, most auctions involve the sale of multiple items to bidders interested in purchasing many items, and valuations differ across bidders.

An implication of the inefficiency result is that there is a class of environments (namely, symmetric private value auctions) in which the symmetric equilibrium of the oft-criticized pay-as-bid auction dominates *all* equilibria of the uniform-price auction in both efficiency *and* seller revenues. However, relaxing the unrealistic symmetry assumption leads to a class of environments where the uniform-price auction outperforms the pay-as-bid auction in both efficiency and revenues. Determining the better pricing rule is necessarily an empirical question.

The practical importance of demand reduction is easily seen in spectrum auctions. Our theorems do not directly apply to the spectrum auctions, since the FCC and others used a simultaneous ascending auction and, often, the licenses are not perfect substitutes. Hence, the analogy between our setting and the spectrum auctions is crude. Still, based on our direct experience in many spectrum auctions around the world, we conclude that demand reduction is of fundamental importance to bidders. Indeed, demand reduction is likely more pronounced in simultaneous ascending auctions than in sealed-bid auctions, since the bidders can propose divisions of the licenses through their early bids. The October 1999 German auction of GSM spectrum illustrates this behavior most clearly. The auction lasted just two rounds—one to propose the split and one to accept it.

Direct evidence of strategic demand reduction was observed in the FCC's Nationwide Narrowband Auction. In round 11, PageNet decided to cut back from bidding on three large licenses to two (Cramton 1995). PageNet felt that, if it continued to demand a third large license, it would drive up the prices on all of the large licenses to disadvantageously-high levels. Hence, it made sense to reduce its demand to two, even though the auction price had not yet reached PageNet's incremental value for a third large license. In making this decision, it was essential for PageNet to anticipate the effect of demand reduction on prices.

Anticipating price movements as a function of one's demand is often guesswork. Still the consequences of guessing wrong can be dramatic, as was illustrated in the August 2000 German auction of third-generation (3G) mobile wireless licenses. After round 127, Deutsche Telekom could have likely brought the auction to a rapid close by reducing its demand from three license blocks to two. Instead, Deutsche Telekom continued bidding for three blocks for some 40 more rounds, ultimately buying the two license blocks that it could have bought earlier, but paying about $2 billion extra.

As in experiments, real-world bidders learn from their mistakes. Three months later, in November 2000, the Austrian 3G auction was held, with essentially the same rules and essentially the same players as the German auction. The starting prices in the Austrian auction were one-seventh of the final German prices (on a population-adjusted basis) and there were sufficiently few bidders that all could be winners if they reduced their demands. All but one of the bidders engaged in demand reduction at the first opportunity and there was a wide presumption that the one holdout, government-owned Telekom Austria, was under severe political pressure to prevent the auction from ending. Even then, demand reduction did well at predicting the outcome: the auction ended in just 14 rounds, at prices only 15% above the low starting prices, and with most participants shading their marginal bids well below their presumed values.

Another important application is in wholesale electricity markets. With only a few exceptions, these are uniform-price auctions conducted daily by the system operator. The theory applies directly to this case, but observe that the strategic response to uniform pricing in a *procurement* auction is *supply* reduction.[9] The incentive to *inflate* bids grows with the quantity supplied, since the higher price is enjoyed on the larger quantity. Wolfram (1998) found compelling evidence of supply reduction in the early years of the UK electricity market. In 2001, the UK switched from uniform pricing to pay-as-bid pricing. More recently, Borenstein, Bushnell, and Wolak (2002) examined why electricity expenditures in California's restructured wholesale market rose from $2 billion in summer 1999 to $9 billion in summer 2000. They found that over one-half of this increase was attributable to market power. In response to the crisis, the California Power Exchange considered switching from uniform to pay-as-bid pricing. Unlike in the UK, the proposal was rejected for the reasons that follow (Kahn et al. 2001).

We present this experience from electricity and spectrum markets to highlight the practical importance of demand reduction, rather than necessarily as an argument against the uniform-price auction. Uniform pricing has several desirable properties, including: (1) it is easily understood in both static and dynamic forms, (2) it is fair in the sense that the same price is paid by everyone, (3) absent

[9] Although these auctions will eventually be two-sided auctions, at present the demand side plays little role, so the auctions are best thought of as procurement auctions in which the system operator purchases energy supplies to meet the anticipated demand.

market power it is efficient and strategically simple ("you just bid what you think it's worth"), and (4) the exercise of market power under uniform pricing favors smaller bidders. While the first three points are commonly made in practice, it is the fourth point that may decisively favor uniform pricing in many practical settings, including some spectrum and electricity markets.

Competition and innovation are often fostered by market designs that encourage the entry and success of small participants. Pay-as-bid pricing disadvantages small bidders: profits depend critically on the bidder's ability to guess the clearing price, and this ability grows with size. In sharp contrast, uniform pricing levels the playing field by weakening the penalty for guessing wrong. At the same time, the current paper has shown that uniform pricing also creates an incentive for large bidders to make room for their smaller rivals.

Milton Friedman (1960, p. 65) recognized the informational leveling effect of uniform pricing in his original proposal: "This alternative, in any of its variants, will make the price the same for all purchasers, reduce the incentive for collusion, and greatly widen the market." We add to Friedman's effect the demand reduction effect, which cuts in the same direction. Moreover, the Treasury auction experiment provides empirical support for the prediction that uniform pricing widens market participation: the five-firm concentration ratio declined by 10 percentage points in auctions that were changed from pay-as-bid to uniform pricing (Malvey and Archibald 1998). The uniform-price auction is not a panacea, since it inevitably yields inefficiency, whereas some alternative multi-unit designs do not (see Vickrey 1961, Ausubel 1997). Nonetheless, good market design should encourage the evolution toward more competitive market structures, and uniform pricing does just that.

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