A Generalized Vickrey Auction

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Abstract

In auction environments where bidders have pure private values, the Vickrey auction (Vickrey, 1961) provides a simple mechanism for efficiently allocating homogeneous goods. However, in environments where bidders have interdependent values, the Vickrey auction does not generally yield efficiency. This manuscript defines a "generalized Vickrey auction" which yields efficiency when bidders have interdependent values. Each bidder reports her type to the auctioneer. Given the reports, the auctioneer determines the allocation that maximizes surplus. The payment rule is the following extension of Vickrey auction pricing: a bidder is charged for a given unit that she wins according to valuations evaluated at the *minimum* signal that she could have reported and still won that unit.

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*The analysis presented here was originally contained in Appendix B of Ausubel (1997). Several colleagues advised me that it was buried and completely missed amid the long paper, and they urged me to repackage it into a more visible form. Finally they have shamed me into removing the appendix from that paper, and writing a separate manuscript.

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INTRODUCTION.

In auction environments where bidders have pure private values, the Vickrey auction (Vickrey, 1961) provides a simple mechanism for efficiently allocating M identical objects. Quite straightforwardly, bidders simultaneously and independently submit up to M bids each; and the M highest bids win. More sophisticatedly, the payment rule is that, if bidder i is to be assigned k objects, then she is charged the k^{th} highest rejected bid (submitted by another bidder) for her first unit, the $(k–1)^{st}$ highest rejected bid for her second unit, … , and the highest rejected bid for her k^{th} unit. As is well known, it then becomes a (weakly) dominant strategy for each bidder to submit bids equaling her true marginal values, yielding efficiency, when bidders have diminishing pure private values.

However, in environments where bidders have interdependent values—meaning that one bidder's value depends on another bidder's signal—the Vickrey auction as defined in the previous paragraph does not generally yield efficiency. While efficiency obtains in single-object environments where bidders are completely symmetric (Milgrom and Weber, 1982) and in two-bidder auctions generally (Maskin, 1992), the Vickrey auction does not generally yield efficiency in single-object environments with three or more asymmetric bidders (Maskin, 1992), nor in multiple-object environments with symmetric bidders (Ausubel, 1997). For the case of a single object where bidders have interdependent values, Maskin (1992) defined a "modified second-price auction" which extends the standard second-price auction to yield efficiency. In the same spirit, in this manuscript, we shall generalize Maskin's approach by defining a "generalized Vickrey auction" for multiple identical objects which yields efficiency when bidders have interdependent values.

The three papers most closely related to the current manuscript are Dasgupta and Maskin (forthcoming), Perry and Reny (1999), and Jehiel and Moldovanu (1999). The first two papers differ from the current one in their basic informational perspective: they assume that the mapping from signals to valuations is commonly known by bidders, but not known by the auctioneer; here it is assumed that the mapping from signals to valuations is commonly known by bidders *and the auctioneer*. Dasgupta and Maskin's work—contemporaneous with Ausubel (1997, Appendix B), the original version of the current manuscript—provides an extremely general, but rather complicated procedure: each bidder communicates her valuation, as a function of each possible realization of all other bidders' valuations, and the auctioneer computes fixed points. Perry and Reny's work—subsequent to Ausubel (1997, Appendix B)—restricts attention, as here, to the case of homogeneous objects and diminishing returns, and then obtains a two-round bidding procedure which is less computationally intensive for the auctioneer. In the first round, bidders simultaneously submit bids, which become public information and which are fully revealing of the bidders' signals. In the second round, each bidder i submits bids $\{b_{ik}^{jl}\}$ representing what bidder i

would submit in a two-bidder, second-price, single-object auction for the k^{th} unit of bidder i versus the l^{th} unit of bidder j. Bidders may need to submit a fairly voluminous collection of bids, as j runs through all bidders ($j \neq i$) while k and l run through all $k + l \leq M + 1$, where M is the number of objects. However, the auctioneer is merely required to pick out the high bids.

The current manuscript, by assuming that the auctioneer also knows the payoff structure, is instead able to get by with a very simple direct mechanism. Each bidder reports her type, and the auctioneer then determines an efficient allocation and payments reminiscent of Vickrey's rule. This makes the mechanism quite intuitive and transparent, and the analysis quite short and simple. Of course, their procedures and mine must be outcome-equivalent.

The stronger informational requirements placed on the auctioneer than in Dasgupta-Maskin and Perry-Reny can perhaps be defended by embedding the direct mechanism as the second stage of a two-stage procedure. In the first stage, each bidder reports the mapping from signals to valuations both for herself and all other bidders. If the bidders make consistent reports, then the auctioneer proceeds to carry out the generalized Vickrey auction; if the bidders make inconsistent reports, then the auctioneer sends everybody home empty-handed.

Jehiel and Moldovanu's work—also subsequent to Ausubel (1997, Appendix B)—shows that with multi-dimensional signals, an efficient direct mechanism is impossible. This is consistent with the current manuscript, as I assume a single-dimensional signal space. For the case of a single-dimensional signal space—and under the hypothesis that each bidder's utility is a linear function of her own and other bidders' signals—Jehiel and Moldovanu extend the direct mechanism of this manuscript to yield efficient outcomes in environments with allocative externalities (i.e., unlike in the other cited papers, they allow each bidder's utility to also depend on the assignments to *other* bidders).

The current manuscript—as well as the three related papers—set efficiency as the sole objective. In the real world, sellers often set reserve prices in auctions. It then becomes an interesting question whether it is possible to extend the selling procedure herein so as to be constrained-efficient subject to the reserve price (i.e., to efficiently assign all objects that are sold, but to inefficiently withhold some of the objects from the market). At the same time, this manuscript (and the three related papers) assumes that payoffs are realized without the possibility for further trade in the auctioned item following the conclusion of the auction. In the real world, agents often engage in post-auction resale. It also becomes an interesting question whether it is possible to embed the efficient equilibrium of the auction into the larger game consisting of an auction round followed by a resale round. Both of these problems are affirmatively solved in Ausubel and Cramton (1999).

THE MODEL AND THE RESULTS.

A seller has a quantity M of a homogeneous good to sell to n bidders, $N \equiv \{1,\ldots,n\}$. The good may be assumed to be either in discrete units or perfectly divisible, with little effect on the analysis. In the discrete case, each bidder i can consume any quantity $q_i \in \{0, 1, \ldots, M\}$. In the perfectly-divisible case, each bidder i can consume any quantity $q_i \in [0,M]$. Let $q \equiv (q_1,\ldots,q_n)$, and let $Q \equiv \{q \mid \sum_i q_i \leq M\}$ be the set of all feasible assignments. Each bidder's value for the good may depend on the private information of all the bidders. Let $t_i \in T_i \subset [0,t_i^{\max}]$ be bidder i's type (i's private information), $t \equiv (t_1,\ldots,t_n) \in T \equiv T_1 \times \cdots \times T_n$, and $t_{-i} \equiv t \sim t_i$. (Type may be discrete or continuous.) A bidder's value $V_i(t,q_i)$ for the quantity q_i depends on her own type t_i and the other bidders' types t_{-i}. A bidder's utility is her value less the amount, X_i, she pays: $V_i(t,q_i) - X_i$. Let $v_i(t,q_i)$ denote the marginal value for bidder i, given the vector t of types and quantity q_i. This is defined so that, in the discrete case, $V_i(t,q_i) = \sum_{k=1}^{q_i} v_i(t,k)$, and in the perfectly-divisible case, $V_i(t,q_i) = \int_0^{q_i} v_i(t,y)\,dy$. We make the following two assumptions on $v_i(t,q_i)$:

Value monotonicity. For all i, t_i, t_{-i}, and q_i, $v_i(t,q_i) \geq 0$, $v_i(t,q_i)$ is strictly increasing in t_i, $v_i(t,q_i)$ is weakly increasing in t_j ($j \neq i$), and $v_i(t,q_i)$ is weakly decreasing in q_i.

Value regularity. For all i, j, q_i, q_j, t_{-i}, and $t_i' > t_i$, $v_i(t_i,t_{-i},q_i) > v_j(t_i,t_{-i},q_j) \Rightarrow v_i(t_i',t_{-i},q_i) > v_j(t_i',t_{-i},q_j)$ and $v_i(t_i',t_{-i},q_i) < v_j(t_i',t_{-i},q_j) \Rightarrow v_i(t_i,t_{-i},q_i) < v_j(t_i,t_{-i},q_j)$.

Value monotonicity implies that types have a natural order, and that bidders have (weakly) diminishing marginal valuations. Observe that, without diminishing marginal valuations, the standard Vickrey auction does not yield efficiency even with pure private values. Value regularity is effectively a single-crossing property: it implies that an efficient assignment rule may be selected so that each bidder i's quantity is weakly increasing in t_i. Value regularity holds if an increase in bidder i's type raises i's marginal value at least as much as that of any other bidder. Without value regularity, Perry and Reny (1999) show that there may not exist any efficient mechanism.

Let $q^*(t) \equiv (q_1^*(t),\cdots,q_n^*(t))$ denote an *ex post* efficient assignment rule for the M objects, i.e., $q^*(t)$ maximizes $\Sigma_i V_i(t,q_i(t))$ subject to $\Sigma_i q_i(t) \leq M$, for all type realizations $t \equiv (t_1,\ldots,t_n) \in T$. (When the efficient assignment is not unique due to flat regions in the aggregate demand curve, $q^*(t)$ is chosen so that each $q_i^*(t)$ is weakly increasing in t_i.) Given efficient assignment rule $q^*(t)$, let us define:

$$\hat{t}_i(t_{-i},y) = \inf\left\{ t_i \mid q_i^*(t_i,t_{-i}) \geq y \right\}. \tag{1}$$

Thus, $\hat{t}_i(t_{-i},y)$ is the minimum report that bidder i can make and still receive at least y units in the efficient allocation, in the event that her opponents report t_{-i}. Finally, it is useful to define $v_{-i}(t,q_{-i})$ as the marginal value of the $(q_{-i})^{\text{th}}$ unit to bidders $-i$ (and given that the units are allocated efficiently among bidders $-i$).

The generalized Vickrey auction is now defined to be the direct mechanism in which objects are assigned according to $q^*(t)$ and the payment rule is defined as the following extension of Vickrey auction pricing: bidder i pays the $\left(q_i^*(t)+1-k\right)^{\text{th}}$ highest rejected value (other than her own) for her k^{th} object where, *crucially*, values are evaluated for this calculation using $\hat{t}_i(t_{-i},k)$ as the signal for bidder i and using t_{-i} (the vector of actual reports) as the signals for bidder i's opponents.[1]

Observe that this static mechanism has the same general flavor as the Vickrey auction. Any bidder's submitted bid does not determine the price she pays (conditional on winning the object), since: (1) à la Vickrey, her payment is determined only by the opportunity cost of providing her with the object; and (2) in computing the opportunity cost, the bidder's actual reported signal is not used, but rather the lowest signal which would enable her to win the object.

More formally, we define:

DEFINITION 1. *Given any efficient assignment rule $q^*(t)$ such that $q_i^*(t)$ is nondecreasing in t_i for each i, the **generalized Vickrey auction** is the direct mechanism in which bidders simultaneously report their types and each bidder i is assigned $q_i^*(t)$ units and is charged a payment $X_i^*(t)$ computed by:*

$$X_i^*(t)=\sum_{k=1}^{q_i^*(t)} v_{-i}(\hat{t}_i(t_{-i},k),t_{-i},M+1-k), \tag{2}$$

in the case of discrete units, and computed by:

$$X_i^*(t)=\int_0^{q_i^*(t)} v_{-i}(\hat{t}_i(t_{-i},y),t_{-i},M-y)\,dy, \tag{3}$$

in the case of perfectly-divisible units.

We easily have the following theorem:

THEOREM 1. *For any valuation functions $v_i(t,q_i)$ satisfying value monotonicity and value regularity, and for any efficient assignment rule $q^*(t)$ such that $q_i^*(t)$ is nondecreasing in t_i for each i, the generalized Vickrey auction has sincere bidding as an ex post equilibrium.*

PROOF. Since $q^*(t)$ has the property that $q_i^*(t)$ is nondecreasing in t_i for each i, $\hat{t}_i(t_{-i},y)$ defined by Eq. (1) is nondecreasing in y. For the case of perfectly-divisible units, substituting Eq. (3) into the expression, $V_i(t,q_i)-X_i$, for bidder i's utility yields the following integral for bidder i's utility from reporting her type as t_i' when her true type is t_i and the other bidders' true and reported types are t_{-i}:

$$U_i(t_i' \mid t_i,t_{-i})=\int_0^{q_i^*(t_i',t_{-i})}\left[v_i(t_i,t_{-i},y)-v_{-i}(\hat{t}_i(t_{-i},y),t_{-i},M-y)\right]dy. \tag{4}$$

[1] In this paragraph, for the case of *perfectly-divisible* units, bidder i then pays the $\left(q_i^*(t)-y\right)^{\text{th}}$ highest rejected value (other than her own) for her y^{th} object.

Observe that the integrand of Eq. (4) is independent of t_i', bidder i's reported type; t_i' enters into Eq. (4) *only* through the upper limit on the integral. Moreover, by value monotonicity, the integrand of Eq. (4) is nonnegative for all $y \leq q_i^*(t)$ and is nonpositive for all $y \geq q_i^*(t)$. Hence, $U_i(t_i' \mid t)$ is maximized when the upper limit on the integral equals $q_i^*(t)$, which is attained by sincere bidding.

For the case of discrete units, the argument is analogous. ■

In the case of a perfectly-divisible good, the expression taken by the payment rule of the Vickrey auction becomes still simpler if the type space is continuous, the valuation functions $v_i(t,q_i)$ are continuous for each i, and if the zero types have zero marginal valuation for the good. In that event, $q_i^*(0,t_{-i})$ may always be taken to be zero, and for $y > 0$, we have that $\hat{t}_i(t_{-i}, y)$ exactly satisfies:

$$v_{-i}(\hat{t}_i(t_{-i}, y), t_{-i}, M - y) \ = \ v_i(\hat{t}_i(t_{-i}, y), t_{-i}, y)\,. \tag{5}$$

Consequently, we immediately have:

PROPOSITION 1. *For the case of continuous types and a perfectly-divisible good, consider any valuation functions $v_i(t,q_i)$ satisfying value monotonicity, value regularity, continuity, and $v_i(0,t_{-i},q_i) = 0$, for all i, t_{-i} and q_i, and any efficient assignment rule $q^*(t)$ such that $q_i^*(0,t_{-i}) = 0$ and $q_i^*(t)$ is nondecreasing in t_i for each i and t_{-i}. Then the generalized Vickrey auction has the payment rule:*

$$X_i^*(t) = \int_0^{q_i^*(t)} v_i(\hat{t}_i(t_{-i}, y), t_{-i}, y)\, dy\,. \tag{6}$$

Finally, observe given Eq. (6) that, under the assumptions of Proposition 1, Eq. (4) reduces to:

$$U_i(t_i' \mid t_i, t_{-i}) = \int_0^{q_i^*(t_i', t_{-i})} \left[v_i(t_i, t_{-i}, y) - v_i(\hat{t}_i(t_{-i}, y), t_{-i}, y) \right] dy. \tag{7}$$

Eq. (7) has an eminently simple interpretation, in close keeping with the traditional mechanism-design literature. Bidder i is precisely permitted to retain her "informational rents": her value for the y^{th} unit is $v_i(t_i, t_{-i}, y)$; she is required to pay only $v_i(\hat{t}_i(t_{-i}, y), t_{-i}, y)$, which would be exactly her value if she were just the minimal type who is assigned a y^{th} unit.

It is interesting to observe that the equilibrium of the generalized Vickrey auction is not only a Bayesian-Nash equilibrium, but also an *ex post* equilibrium: given that bidder i knows the announcement, t_{-i}, that bidders $-i$ will make (and believes the announcement), bidder i still finds it a best response to announce her true type. Given that bidders will not possess dominant strategies in an environment with interdependent values, this is about the strongest result we can hope for. Moreover, since this is an *ex post* equilibrium, observe that the outcome is independent of the joint distribution of types.

Finally, let us relax the informational requirements placed on the auctioneer by specifying a two-stage revelation procedure in which the auctioneer need not know the mapping from signals to valuations. (The mapping still needs to known by all the bidders.) In the first stage, each bidder simultaneously reports the mapping from signals to valuations both for herself and all other bidders. If all of the bidders' reports agree, then the auctioneer proceeds to calculate and carry out the generalized Vickrey auction for the reported mapping from signals to valuations; if the bidders' reports do not agree, then the auctioneer sends everybody home with zero goods assigned and zero payments. Since every bidder's interim payoff in the generalized Vickrey auction is nonnegative (in fact, every bidder's *ex post* payoff is nonnegative), we easily have:

THEOREM 2. *For any valuation functions $v_i(t,q_i)$ satisfying value monotonicity and value regularity, and for any efficient assignment rule $q^*(t)$ such that $q_i^*(t)$ is nondecreasing in t_i for each i, the two-stage procedure has truthful reporting in the first stage and sincere bidding in the second stage as an equilibrium.*

CONCLUSION: COMPARISON WITH ASCENDING-BID AUCTIONS.

For some symmetric models with interdependent values, there exist ascending-bid auction procedures which also yield efficient allocations. For a single indivisible object, this is provided by the English auction (Milgrom and Weber, 1982). For M objects and unit demands, this is provided by an ascending-clock auction which ends at the moment the $(M{+}1)^{st}$ bidder drops out. For M objects and flat demands, this is provided by my efficient ascending-bid auction design (Ausubel, 1997). It is interesting to now observe how the outcome of the generalized Vickrey auction compares with the outcome of the efficient ascending-bid auction under these circumstances.

With a single indivisible object, the winner's payment in the generalized Vickrey auction equals $v_{-i}(\hat{t}_i(t_{-i},1),t_{-i},1)$, coinciding with the payment in the English auction. Thus, the equilibria are outcome-equivalent. However, if there are $M \geq 2$ objects and if efficiency requires that a positive quantity of objects be awarded to two or more of the bidders, then the outcomes differ in a subtle way. In the generalized Vickrey auction, all of the private signals have been revealed to the mediator, and the payment is then allowed to depend on all $(n-1)$ private signals of the other bidders. By way of contrast, in the efficient ascending-bid auction designs, some or all of the objects are awarded at a time when two or more bidders remain in the auction. Consequently, the payment then depends on $(n-2)$ or fewer of the private signals of the other bidders. By the same logic as in Milgrom and Weber (1982), in environments where the bidders' signals are strictly affiliated, the generalized Vickrey auction uses more private signals and hence yields higher expected revenues. Indeed, as shown by Perry and Reny (1999), it yields an upper bound for the expected revenues of an efficient ex post equilibria, and thus, it is the appropriate benchmark for comparing the revenues of the efficient ascending-bid auction designs.

At the same time, the generalized Vickrey auction has a serious disadvantage relative to the efficient ascending-bid auctions. To paraphrase Maskin (1992, p. 127, footnote 3), the reader should notice that the rules of the generalized Vickrey auction are *defined* in terms of the mappings from signals to valuations. That is, the auction designer must know these mappings (à la Theorem 1), a demanding requirement, or ask them and obtain consistent responses (à la Theorem 2), still a reasonably implausible task. By contrast, the designer can remain ignorant of the mappings if he uses the efficient ascending-bid design, a distinct advantage that may well more than offset the theoretical disadvantage in expected revenues.

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