The Ascending Auction Paradox

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Abstract

We examine uniform-price, multiple-unit auctions that are dynamic rather than sealed-bid, auctions where the price ascends, giving bidders the chance to observe and respond to the bidding. In this dynamic setting, subgame perfection can be exploited to refine the set of equilibria. Restricting attention to ascending auctions where two bidders with complete and perfect information alternate placing bids until the market clears, we show that in the *unique* equilibrium each bidder reduces her quantity to the market-clearing level at her first opportunity, when the price has not yet risen. For much the same reason that Rubinstein bargainers choose to trade immediately even though they could alternate making offers indefinitely, a bidder in an ascending auction calculates where the auction would go, and then uses backward induction to find an offer that is acceptable to her rival at the start of the auction.

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1 Introduction

When a seller has many units to sell, she may decide to bundle them together and auction the lot. But when the units for sale may be worth more than any one buyer can afford, or if the seller is a government who worries about creating monopoly power by putting all of a resource into one firm's control, the seller may wish to divide the entire quantity before selling it. Game theorists have recently made strides in understanding multi-unit auctions, but most of this work has assumed sealed bids. Open auctions, at least for the single-unit auctions that were traditionally studied, have several advantages over closed (sealed-bid) auctions. Auction houses, like Christie's and Sotheby's, typically use the open or *English* auction, where the auctioneer raises the price until only one bidder—the winner—is willing to purchase the item. When the value of the item for sale is uncertain to the bidders who may only have noisy estimates, open auctions can stimulate bidding if bidders can infer information about the other bidders' estimates for the item. When multiple units are for sale, the advantages of using an open auction may be even more important. Bidders may be able to process the information revealed in the auction, and then update their value estimates and bidding strategies for all of the units. Additionally, as in the case of the FCC's spectrum auctions where the units for sale are heterogeneous, open multi-unit auctions provide bidders the flexibility to pursue contingent strategies.

However, we will show that in uniform-price ascending auctions for multiple units, bidders have strong incentives to reduce the quantities they demand to prevent the price from rising. We will show how strong these incentives can be in a simple model with complete and perfect information, where bidders have constant valuations over the quantity for sale. In our model, a seller with some perfectly divisible quantity auctions it to two bidders using an *ascending auction*. The auctioneer raises prices, and alternates asking first one bidder how much quantity she demands at the current price, then asking the other bidder how much quantity she demands at the next price. The auctioneer continues to raise price until there is no excess demand. As in a single-unit English auction, when the auction ends, bidders pay their bids for the units they win. In this setting, we show that there is a unique subgame equilibrium outcome, and in this outcome bidders reduce to market clearing quantities at the start of the auction. The paradox is that in the ascending auction prices never ascend—settlement is immediate.

The game analyzed in this paper is the format that a seller might initially envision using in an online auction of multiple identical objects, such as U. S. Treasury bills, shares of stock in IPO's, emissions permits, and industrial parts. Indeed, in the aftermath of the Salomon Brothers' scandal, it was proposed in the *Joint Report on the Government Securities Market* (1992) that U. S. Treasury securities be sold with exactly this auction:

> [R]egistered dealers and other major market participants would have terminals that are connected by telephone line (with appropriate security) to a central computer. The auction would begin with the Treasury announcing an opening yield somewhat above the yield at which the security is quoted in when-issued trading. All interested parties would then immediately submit tenders electronically for the quantity of securities they would be willing to purchase at that yield
>
> Once all bids were submitted, the resulting total volume of bids at this yield would be announced. [If the bids exceeded the quantity issued] the yield would then be reduced, perhaps by one basis point, and the bidding process repeated. Bidding would proceed in successive rounds… with decreasing yields until the volume demanded was smaller than the size of the issue. All participants who bid at the closing yield would receive awards, but at the next higher yield (U.S. Department of the Treasury, Securities and Exchange Commission, and Board of Governors of the Federal Reserve System, 1992, page 15).

We show in this paper that the auction format has potentially disastrous consequences. Bidders have the incentive to divvy the quantity at the opening price, to end the auction before profit margins are eroded.

The "paradox" in our title is intended to evoke Selten's (1978) chain store paradox. Selten's paradox is that the unique backward induction solution to the chain store game is so at odds with intuition. Backward induction requires that in every period a different potential entrant opens a store, and then the chain store monopolist accommodates, rather than deters, this entrance. Intuition predicts that the monopolist should forcefully deter entrance in the early rounds to discourage future entrants, thereby increasing the monopolist's payoff. Selten candidly words the paradox:

> If I had to play the game in the role of [the monopolist], I would follow the deterrence theory. I would be very surprised if it failed to work… My experience suggests that mathematically trained persons recognize the logical validity of the induction argument, but they refuse to accept it as a guide to practical behavior (pages 132-133).

Selten's paradox, then, is a critique of backward induction. In our paper, we also derive our result with a heavy dose of backward induction. Unlike the chain store contest, however, the unique subgame perfect equilibrium outcome in the auction game is quite generous to all of the players. In the case of ascending auctions, it is the backward induction that we believe makes our prediction reasonable. Bidders have time to contemplate the detrimental effects of aggressive bidding. A bidder's aggressiveness will not only raise the price for other bidders, but will also raise the price for units she expects to win. It seems likely that conducting the auction dynamically helps the bidder see that splitting the market at a lower price can be more profitable than bidding sincerely. It is well-documented that bidders conform to predicted behavior in open settings where they have time to learn what are the best strategies but that bidders deviate from predicted behavior in static, sealed-bid settings. Kagel, Harstad, and Levin (1987) and Kagel and Levin (1993) show that bidders are more likely to play the dominated strategy of bidding above their values in a single-unit second price auction than they are in an English auction. Further, Kagel and Levin (1997) show that bidders are more likely to adhere to the demand reduction equilibrium in the ascending auction than they are in the sealed-bid auction in a multi-unit auction. Alsemgeest, Noussair, and Olson (1998) provide evidence that bidders with independent private values who each demand two units often demand reduce in ascending auctions, dropping to one unit at prices lower than their valuations. The authors offer the possible explanation that the demand reduction "may have been an effort to cooperate to lower price" (page 95). Because the benefits of this type of cooperation can be predicted with backward induction, a concept relevant to open auctions, it is our hypothesis that open auctions can perform very poorly.

An apt title for a literature review is: Uniform-Price Auctions and What Can Go Wrong. What goes wrong stems from the uniform pricing. Once a bidder realizes that by bidding aggressively she risks raising the price on all the units she will win, she may prefer to bid less aggressively to keep the price low. That is, in uniform-price auctions a bidder often trades quantity for price: less quantity but at a higher profit margin. The literature describes how this incentive can lead to low revenues or inefficiency. An early example is Wilson (1979). Wilson assumes there is a perfectly divisible good for sale that has a common value to the bidders. He shows in examples with complete and incomplete information that in a sealed-bid, uniform-price auction, bidders can achieve low price equilibria, even when the number of bidders is large. Back and Zender (1993) consider a model similar to Wilson's, where a seller uses a uniform-price auction to sell a common-value good. They show that there exist equilibria that achieve whatever constant reserve price the seller announces. Pavan, Licalzi, and Gilli (1998) extend this model by showing how the seller can set an upward-sloping reserve price schedule to partially protect herself from the self-enforcing strategies described by Back and Zender. With two discrete goods, Noussair (1995), Engelbrecht-Wiggans and Kahn (1998), and Katzman (1997b) show that with independent private values, each bidder shades her bid on the second unit she demands in sealed-bid, uniform-price auctions. In particular, Engelbrecht-Wiggans and Kahn emphasize the conditions under which two-object, sealed-bid auctions yield "zero-price" equilibria, the sealed-bid analogue of the subgame perfect equilibria in our paper. Ausubel and Cramton (1996) show that because of the incentive to demand reduce, or to submit bid functions that lie below demand functions, it holds quite generally that there do not exist efficient equilibria in uniform-price auctions where bidders have either independent or affiliated values.

There has been much less work describing the incentives bidders have in open uniform-price auctions. Cramton (1998) discusses some of the virtues and pitfalls of using an ascending uniform-price auction over a sealed-bid uniform price auction. Engelbrecht-Wiggans and Kahn (1999) show that in an independent private values model with heterogeneous units, there exist low price equilibria in a simultaneous-move ascending auction where bidders split the available units for sale in the first round, and then enforce this split by threatening to punish bidders who interlope on another's units. This kind of punishment strategy is common in the FCC's spectrum auctions (see Cramton and Schwartz, 1998a, b). Menezes (1996) shows that with complete information and private values, there exist low price equilibria in a simultaneous-move, ascending auction. However, Menezes is only able to prove that low prices are mandatory under the assumption that equilibria are "Pareto perfect." In using a Pareto refinement, he *assumes* that, if a bidder is willing to accept some payoff in some round, then the bidder must be willing to accept any quantity that yields her at least as much payoff in earlier rounds. By contrast, we prove that it is unnecessary to impose the Pareto refinement; low prices are a necessary consequence of subgame perfection alone.

In our paper, we need only appeal to the subgame perfection refinement to derive the uniqueness of the equilibrium path, a path that calls for the players to split the available quantity at the start of the auction before prices rise. We describe our model in Section 2. The model implies a few results about behavior in the auction; we show these results in Section 3. In Section 4, we study the auction game where bidders are permitted to bid for any share of the quantity in any round that they bid, regardless of past bidding. In this game, there is a unique subgame perfect equilibrium path, where the first bidder proposes some division of quantity, and the second player accepts. In Section 5, we show that a similar result holds even if we restrict the bidding so that once a bidder places some quantity bid, she can never place a higher bid later in the auction. We view this model as a closer approximation to observed auction mechanisms. This model also is a closer analog to the sealed-bid auction where bidders are required to submit downward sloping demand curves. For this auction, we explicitly solve for the bid the first bidder offers (which the second player accepts). We derive a limit result that says as the bid increment between rounds goes to zero, the first bidder's offer is her value divided by the sum of the two bidders' values. For very small bid increments and values not too dissimilar, this means that the bidders split the available quantity approximately in half. In Section 6, we provide some evidence that our result is not limited to auctions with complete information or with a perfectly divisible good. Our conclusions are in Section 7.

2 Model

A seller has some continuously divisible quantity of assets for sale. We normalize this quantity to $Q = 1$. In the literature this is called a share auction, the interpretation being that bidders compete over what percentage they want. There are two bidders, A and B, who have constant valuations for the entire unit. Denote the bidders' marginal values by V_A and V_B. The bidders payoff functions are linear: $U_i(q, x) = qV_i – x$ if bidder *i* wins q units and pays x. We assume complete information, meaning that both A and B know these payoff functions.

The extensive form of the game is *loosely* inspired by the FCC spectrum auctions, and is common knowledge to the players.[1] Before the auction begins, the seller announces a reserve price P_0, a first bidder 1 (1 = A or B), a second bidder 2 (the other bidder), and quantity restrictions $q_{1,-1} \in (0,1]$ and $q_{2,0} \in (0,1]$. A bidder can neither bid for nor win more than her quantity restriction. So that our auction is not trivial we assume that $q_{1,-1} + q_{2,0} > 1$. Bidding occurs in rounds t = 1, 2, … until the auction ends.[2] The price rises by the constant increment Δ in each round; in round t, the asking price is $P_t = P_0 + t\Delta$. We assume that the auctioneer takes bids first from bidder 1, then from bidder 2, alternating between the bidders until the auction ends.

Alternating Bids: In round 1 and all odd rounds such that the auction has not ended, bidder 1 bids. In round 2 and all even rounds such that the auction has not ended, bidder 2 bids.

We consider two games. In the Unconstrained Eligibility Game, a bidder may bid for any quantity less than her quantity restriction, regardless of her past bidding. In the Monotonic Eligibility Game, a bidder may only place bids that do not exceed her prior bid. Definitions are given below.

Unconstrained Eligibility Game: Bids $q_{i,t}$ (*i* =1, 2 and t ≥ 1) must satisfy the *quantity restriction rule*:

$q_{1,-1} \geq q_{1,t}$ for t = 1, 3, …

$q_{2,0} \geq q_{2,t}$ for t = 2, 4, …

Monotonic Eligibility Game: Bids $q_{i,t}$ (*i* =1, 2 and t ≥ 1) must satisfy the *monotonic eligibility rule*:

$q_{1,-1} \geq q_{1,1} \geq q_{1,3} \geq \dots$

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[1] For a description of the rules and performance of the FCC's spectrum auction, see Ausubel et. al. (1997), Cramton (1995, 1997), McAfee and McMillan (1996), and McMillan (1994).

[2] It will not matter if the auctioneer announces some finite "last round" so long as this last round is sufficiently large—see Assumption 1 in Section 3.

$q_{2,0} \geq q_{2,2} \geq q_{2,4} \geq \ldots$

Definition 1: An offer is *feasible* if it satisfies the rules of the game being considered.

The auction continues so long as there is excess demand. If *i* places a bid $q_{i,T}$ in round $T \geq 2$, such that $q_{i,T} + q_{j,T-1} \leq 1$, then the auction ends. The auctioneer awards the entire unit giving preference to bids placed at the highest price. As in the English auction or the FCC spectrum auctions, bidders must pay the price at which they win quantity. If the auction ends in round $T > 2$, then *i* wins $q_{i,T}$ at price P_T, *j* wins $q_{j,T-1}$ at price P_{T-1}, and *i* wins the remaining quantity $1 - q_{i,T} - q_{j,T-1}$ at price P_{T-2}. If the auction ends in round 2, then the auctioneer awards $q_{2,2}$ to bidder 2 at price P_0 and awards $q_{1,1}$ to bidder 1 at price P_1, returning any remaining quantity to the seller. Observe that the auction cannot end in round 1 before bidder 2 has a chance to bid. We emphasize that bids are real financial commitments—if a bidder places a quantity bid at a given price, it is possible that she may win this quantity at this price.

Next we specify histories, strategies, and the equilibrium concept we will use to analyze this game. Define the history for round 1 as $H_1 = \{(q_{1,-1}, q_{2,0})\}$, where the quantity restrictions $q_{1,-1}$ and $q_{2,0}$ can be thought of as bids that occurred in rounds –1 and 0. For $t > 1$, let h_t be any feasible sequence of bids in rounds –1 through $t - 1$ (where bids in –1 and 0 are the quantity restrictions $q_{1,-1}$ and $q_{2,0}$). Let H_t be the set of all such sequences h_t. For odd rounds t, a round-t behavioral strategy $q_{1,t}$ for bidder 1, the bidder who moves in round 1, is a function that assigns for each $h_t \in H_t$, a probability distribution (on those quantities permitted by the rules of the game we consider). Note that this is a considerable abuse of notation, since we also use $q_{1,t}$ to denote the realization of bidder 1's round-t bid. The vector of each $q_{1,t}$ (t = 1, 3, …) is bidder 1's strategy q_1. Bidder 2's strategy q_2 is defined likewise. A pair (q_1, q_2) induces a probability distribution over outcomes after any history h_t. A subgame perfect equilibrium is a pair (q_1^*, q_2^*) such that neither player can profitably deviate after any history h_t given that the players adhere to the specified strategies after the deviation.[3] In this paper, we will focus on subgame perfect equilibrium (SPE).

Aside: To approximate the auction where bidders name their bids, we have chosen this alternating bid model. The common alternative that many modelers have fruitfully used is the so-called button model, where each bidder depresses buttons to indicate the quantity she bids for as the auctioneer raises prices. Of course, physical buttons need not exist— they are just a literary tool to help explain the

[3] Where by "profitably" we mean that which increases the expected value of utility.

extensive form of the auction. In the irrevocable-exit button models, used to analyze unit auctions by Milgrom and Weber (1982) and multi-unit auctions by Menezes (1996) among others, bidders must bid in every round or be disqualified from the auction.[4] In English auctions like those used by Sotheby's, a bidder can name a bid, and then not bid again until another bidder names a higher price. Often when the bidding is down to two bidders, they will alternate naming bids until only one bidder remains. In English auctions a bidder can jump bid or raise her own bid, but this is usually to send a signal (see Avery, 1998). In the FCC spectrum auctions, it is our experience that bidders usually alternate bidding, one bidder placing a bid on some license in one round, and a rival topping this bid in the next round. When bidders do raise their own bids, it is usually to signal strong interest in the license. We view our alternating bid model as a useful approximation to this type of auction, an approximation that abstracts away from such signaling. Our model is closer in spirit to Milgrom's (2000) model of a multi-unit auction, where there are two relevant prices in the auction, one price if she is the high bidder (her bid) and another price if she is not the high bidder (the high bid plus a bid increment).

3 Preliminaries

Before moving to our main results in the next sections, there are a few preliminaries that result almost immediately from the specification of our model. These preliminaries will enhance the interpretation of later results.

Lemma 1: For either the Unconstrained or Monotonic Eligibility Rule Game, any offer $q_{i,T} > 0$ that ends the auction in round $T > 2$ is strictly dominated by an offer of 0.

Proof: If the auction closes after *i* places a bid $q_{i,T} > 0$, then it must be that $q_{i,T} + q_{j,T-1} \leq 1$. Then *i* wins $q_{i,T}$ at price P_T and $1 - q_{i,T} - q_{j,T-1}$ at price P_{T-2}. In total, *i* wins $1 - q_{j,T-1}$ units, paying P_T for $q_{i,T} > 0$ units. Alternatively, by bidding 0 in round T, *i* still wins $1 - q_{j,T-1}$ units, but pays the lower per unit price P_{T-2} for all of this quantity. ∎

By Lemma 1, we will be able to say that a bidder "accepts" if she ends the auction after round 2, knowing that any optimal strategy insists that she bids 0. Also, by Lemma 1, this auction is uniform-price in nature,

[4] For a description of many types of button auctions, see Bikhchandani and Riley (1991).

meaning that each bidder chooses to win all of her quantity at the same price. We give some definitions below.

Definition 2: If a bidder bids 0 in round $t > 2$, we say that she *accepts* the offer made by her rival in the previous round. If $V_2 > P_0$, we say that bidder 2 *accepts* an offer in round 2 if she ends the auction by bidding the lesser of her eligibility ($q_{2,0}$) and the residual of bidder 1's round 1 bid ($1 – q_{1,1}$). In round t, bidder *i rejects* an offer $q_{j,t-1}$ if she counteroffers $q_{i,t} > 1 – q_{j,t-1}$, so that the auction proceeds to round t+1.

Throughout the paper, we will maintain the following assumption requiring that bid increments are sufficiently small and that values are generic relative to each other and the bid grid.

Assumption 1: $V_A > V_B$ and $V_A \neq P_t$ for t = 1, 2, … The bid increment Δ is small enough so that for some round S when it is B's move, the following holds: $P_S < V_B < P_{S+2} < V_A$. (If rather than allowing an infinite number of rounds, the auctioneer commits to a "last round" L, then L is large enough so that $P_L > V_A$).

Remark: By Assumption 1, A is the high valued bidder. Also by Assumption 1, S is the last round B moves such that the price is below her value. We will use Assumption 1 to construct a last interesting round (S + 2) such that even if the horizon is infinite, play will not continue past this round.

Lemma 2: For any subgame which begins in round S+2, in any SPE bidder B accepts with probability one any $q_{A,S+1} < 1$ made by A in round S+1.

Proof: By accepting $q_{A,S+1} < 1$ in round S+2, B obtains the payoff $[1 – q_{A,S+1}][V_B – P_S]$, which is positive by Assumption 1. Alternatively, by rejecting, B can no longer obtain any quantity at a price less than her value, eliminating the possibility of earning a positive payoff. ❚

Definition 3: For round S+1, any feasible offer $q_{A,S+1} \leq 1$ made by A is *acceptable*, and the supremum of acceptable offers is A's *best acceptable* offer $\overline{q}_{A,S+1}$. For round t ≤ S, a feasible offer $q_{i,t}$ by bidder *i* in round t is defined recursively as:

$$\begin{Bmatrix} \textit{strictly acceptable} \\ \textit{acceptable} \\ \textit{unacceptable} \end{Bmatrix} \quad \text{if } [1 - q_{i,t}][V_j - P_{t-1}] \begin{Bmatrix} > \\ \geq \\ < \end{Bmatrix} \bar{q}_{j,t+1}(h_t, q_{i,t})[V_j - P_{t+1}],$$

where $\bar{q}_{j,t+1}(h_t, q_{i,t})$ is the supremum of acceptable offers *j* can make in round t+1, given the history h_t and offer $q_{i,t}$. Let $\bar{q}_{i,t}(h_t) = \sup\{q_{i,t}: q_{i,t} \text{ is acceptable}\}$. Call $\bar{q}_{i,t}(h_t)$ the *best acceptable offer* bidder *i* can make in round t.

Remark: While *strictly acceptable, acceptable,* and *unacceptable* are recursively defined, their meanings coincide with conventional meanings if $\bar{q}_{j,t+1}(h_t, q_{i,t})[V_j - P_{t+1}]$ is bidder *j*'s continuation payoff if $q_{i,t}$ is rejected. Our work will show this is the relevant continuation payoff, since a bidder will not find it profitable to make unacceptable offers. We will prove that in all SPE, bidders will accept with probability one *strictly acceptable* offers and will reject with probability one *unacceptable* offers.

We next introduce some additional notation, notation that will draw the reader's attention to similarities between our model and Rubinstein (1982) bargaining between two traders who discount payoffs. Suppose in round $t \geq 2$, bidder *i* is deciding between ending the auction in round t and prolonging the auction by making some offer greater than the residual supply. By ending the auction she can obtain the profit margin $V_i - P_{t-2}$ (by Lemma 1). However, by prolonging the auction to the next round, *i*'s profit margin is at most $V_i - P_t$. We define a "discount factor" $\delta_{i,t}$ as the rate at which a bidder's profit margin declines by prolonging the auction's settlement to the next available price.

Definition 4: Let $\delta_{i,t} = \dfrac{V_i - P_t}{V_i - P_{t-2}}$ for t = 2,…

Remark: It follows that the discount factor $\delta_{i,t} < 1$ for t satisfying $V_i - P_t > 0$.

Lemma 3: (decreasing discount factors): If $t \geq 2$ and if $V_i - P_t > 0$, then $\delta_{i,t} > \delta_{i,t+2}$.

Proof: It suffices to show that $[V_i - P_t]^2 > [V_i - P_{t+2}]\,[V_i - P_{t-2}]$, or equivalently, $[V_i - P_t]^2 > [V_i - P_t - 2\Delta]\,[V_i - P_t + 2\Delta]$. This follows since $-4\Delta^2 < 0$. ▌

Lemma 3 says that a player's discount rate decreases over time; that is, a player becomes increasingly impatient as the auction progresses towards the price at which she no longer wants to win any quantity.

With these preliminary results in hand, we are now ready to analyze the Unconstrained Eligibility Game.

4 Unconstrained Eligibility Game

The first game we study is the Unconstrained Eligibility Game, where a bidder may place any quantity bid when it is her turn. A backward induction argument will establish that the unique outcome is that bidder 1 makes some offer that bidder 2 accepts.

Theorem 1: Suppose V_A and V_B satisfy Assumption 1. Then the Unconstrained Eligibility Game has a unique SPE path. On this path, bidder 1 makes her best acceptable offer in round 1, and bidder 2 accepts in round 2.

This theorem is simple to derive using backward induction. In fact, the proof is much like one would prove the uniqueness of a finite-horizon Rubinstein bargaining game with discounted payoffs. The finite horizon in our proof comes from the last interesting period – Lemma 2 above establishes that play will not proceed past round S+2 in any SPE. Then assuming that there is a unique SPE outcome in rounds t+1, …, S+2, each bidder would prefer to keep for herself any surplus the next player would forfeit by not accepting and making her equilibrium offer. Details of the proof are in Appendix 1.

But this game may not be a good approximation to real-world auctions. In some sense, with no eligibility constraints, the first bidder may decide to place a low bid to see if the other bidders will cooperate to achieve a low price. The cost of doing this is negligible since if the other bidders do not cooperate, the first bidder can always bid aggressively later. In the FCC spectrum auctions, which did constrain bidders to use or lose eligibility, serious bidders typically did not reduce their eligibility much at the beginning of the auction. Was this due to the eligibility constraint? In the next section, we analyze the more complicated game where once a bidder reduces her quantity, she forfeits the ability to bid for higher quantities later in the auction.

5 Monotonic Eligibility Game

In this section, we analyze the Monotonic Eligibility Game, which requires a bidder to forfeit eligibility if she reduces her quantity bid. This is analogous to irrevocable-exit button auctions, or the use it or lose it rules in the FCC spectrum auctions. With the monotonic eligibility rule imposed, a bidder may be reluctant to reduce immediately to the market clearing level since she has no guarantee that the next bidder will cooperate, and any reduction in eligibility may mean losing bargaining strength. Intuition might suggest that bidders alternate reducing their eligibilities by small amounts until the auction ends. This outcome would resemble the equilibrium in Admati and Perry's (1991) contribution game, where the players alternate making small contributions towards a joint project until the sum of the contributions exceeds the project's cost.

However, just as in the Unconstrained Eligibility Game, the bidder who moves first immediately reduces her quantity to the market clearing quantity, and the next player ends the auction. We state this result precisely in Theorem 2.

Theorem 2: Suppose V_A and V_B satisfy Assumption 1. Then the Monotonic Eligibility Game has a unique SPE path. On this path, bidder 1 makes her best acceptable offer in round 1 and in round 2, bidder 2 accepts. If $q_{1,-1} \geq \frac{1-\delta_{2,2}}{1-\delta_{1,3}\delta_{2,2}}$ and $q_{2,0} \geq \frac{1-\delta_{1,3}}{1-\delta_{1,3}\delta_{2,2}}$, then bidder 1's round 1 offer is $\frac{1-\delta_{2,2}}{1-\delta_{1,3}\delta_{2,2}}$.

Proof: The proof is in Appendix 2.

The proof takes advantage of two things: backward induction and the perfect divisibility of the unit. Lemma 2 above shows that play will not proceed past round S+2. From this point, we work backward. In every round, we prove that a bidder prefers to either accept or make some acceptable counteroffer rather than make some unacceptable offer. The reason is that when it is a bidder's turn to bid, she has a first mover advantage. This advantage arises because if the next bidder refuses an offer, she will raise the price on the quantity that she will win. Since quantity is perfectly divisible, a bidder can offer her rival just enough quantity to induce the rival to settle immediately at a lower price rather than later at a higher price. The first mover advantage is akin to that in Rubinstein bargaining. In Rubinstein bargaining the cost of delaying is one period's discounting (or one period's cost of bargaining). In our model and in Rubinstein's model, each player takes advantage of the other's impatience when determining her best acceptable counteroffer.

Though the logic of the proof is straightforward, the proof is far from trivial. The eligibility rule means that there are subgames when a bidder is not allowed to make the highest offer the next bidder will accept. In Appendix 2 (Claim 3), we show that by making an acceptable offer in round t, a bidder will not have the eligibility to make the highest acceptable offer in round t + 2. It is this result, that makes the round 1 offer we state in Theorem 2 resemble the initial offer in the Rubinstein (1982) bargaining game where the players have constant discount rates. For in the Rubinstein game with an infinite horizon and stationary payoffs, a player's optimal offer in one period is identical to her optimal offer two periods later.

To diminish the first mover advantage, we next take the limit as the price between rounds goes to zero.

Corollary: For n = 1, 2, …, define $G_n(\Delta_n)$ as the Monotonic Eligibility Game with bid increment Δ_n = 1/n. Then as $n \to \infty$, the initial offer in G_n goes to $\frac{V_1 - P_0}{V_1 + V_2 - 2P_0}$.

The corollary implies, the bidder who moves first wins more quantity the higher her value, and wins less the higher her opponent's value. For values not too dissimilar and for $P_0 = 0$, the bidders each win approximately half the quantity.

6 Discrete Units and Incomplete Information

The uniqueness that we derived in Theorems 1 and 2 may result from our assumptions of complete and perfect information, the ripe setting for a backward induction solution. But the idea that bidders accommodate their rivals in the ascending auction applies more generally. With incomplete information, a bidder must ask herself: Should I make gestures to end the auction now with the price very low, or should I bid aggressively in hope that my opponent's values are low or that my opponent will accommodate my demand? In this section, we will suggest with an example that the answer to this question is in favor of keeping the price low. We will show that the result that bidders are able to coordinate a division of the available quantity at low prices is not merely an artifact of our assumptions of complete information and perfect divisibility (though these assumptions permitted a rigorous proof in a generic setting). We will give an example where even though a seller uses an ascending auction to sell two discrete units to incompletely informed buyers, the buyers still forgo bidding competitively in favor of each winning one unit at the lowest allowable prices. More importantly, we show that this demand reduction occurs for the same reason: bidders predict the (expected) consequences of forcing the price up, and prefer to settle at low prices.

Assume the seller has two *discrete* units to sell to our bidders A and B, who each have constant marginal values and a capacity for both units. Let these valuations be V_A and V_B. We will make different assumptions about the distributions these values are drawn from. In this section, we require bids to be discrete, $q_{i,t} \in \{0, 1, 2\}$, and we require that bids satisfy the monotonic eligibility rule described in Section 2.

The first example shows that bidders coordinating a division of the available quantity early in the auction does not depend on our perfect divisibility assumption maintained in Sections 3 through 5.[5] Our second example shows that this coordination does not require complete information.

Example 1 (Degenerate Distribution): Suppose A's valuation equals V_A with probability one and B's valuation equals V_B with probability one. Suppose that (*i*) $V_A - V_B > \Delta$ and (*ii*) $V_A - P_1 > 2[V_A - (V_B - \Delta)]$.[6] Then in any SPE where bidders use weakly undominated strategies, each bidder wins one unit, the auction ends before reaching round 4, and revenues are at most $P_1 + P_2$.

Proof (Sketch): The following can be shown to be weakly dominated strategies:

- a player bidding for zero units at a price below her value
- a player bidding on more than zero units is more than her value.

Consider subgames beginning in round t where it is A's move, where A's eligibility is two units (meaning that A has not yet made a bid for less than two units) and B's eligibility is one unit, and where t satisfies: $[V_A - P_{t-2}] > 2[V_A - (V_B - \Delta)]$. Bidder A's best response is to accept, winning one unit at price P_{t-2}. This is the lowest price that A can win one unit. For A to win two units, B would have to bid zero

[5] Our first example corresponds to the equilibrium Katzman (1997a) derives for a sealed-bid, uniform-price auction with complete information given the two bidders' values satisfy his condition (1a*), which states the high valued bidder's value for her second object is not large relative to the low-valued bidder's value for a second object.

[6] Assumption (*ii*) requires that A's valuation is small enough that she prefers winning 1 unit at a low price over winning two units at a price high enough to bump B out of the auction.

units, but she will not do this until the round occurs where price exceeds her value. When B bids zero units, A will win two units at the prior price, which is at least $V_B – \Delta$. Thus, A's payoff from winning two units is at most $2[V_A – (V_B – \Delta)]$, less than A would get by reducing to one unit in round t.

Consider subgames t where it is B's move, where A and B are each eligible for two units, where the price is below B's value, and where t satisfies: $[V_A – P_{t-1}] > 2[V_A – (V_B – \Delta)]$. By the argument above, if B bids one unit at the current price, then A will accept in the following round. This gives B her highest possible payoff given that A will not bid zero units at prices below V_B. Since B strictly prefers to win one unit at the current price to winning one unit at any higher price, B will bid two units only if A's strategy is to bid one unit in the next round. Otherwise, B will bid one unit in the current round.

Consider rounds t where it is A's move, where A and B are each eligible for two units, where $P_{t+1} < V_B$, and where t satisfies: $[V_A – P_t] > 2[V_A – (V_B – \Delta)]$. Bidder A can win one unit at the current price by bidding one unit in round t, after which B will accept in round t+1. Because A strictly prefers winning one unit at the current price to winning two units at price $V_B – \Delta$, A will only bid two units if B's strategy is bid one unit in round t+1 (so that A can accept in round t+2); otherwise, A will bid one unit in the current round.

Finally, consider the first round. Suppose it is A's move. Since $V_A – P_1 > 2[V_A – (V_B – \Delta)]$, then by the above argument A will bid one unit in round 1 and B will accept in round 2, or A will bid two units in round 1, B will bid one unit in round 2, and A will accept in round 3. In either case, A will win one unit at price P_1 and B will one unit at no more than P_2. Alternatively, suppose it is B's move in round 1. Then since $V_A – P_0 > 2[V_A – (V_B – \Delta)]$, B will either bid one unit in round 1 and A will accept in round 2, or B will bid two units in round 1, A will bid one unit in round 2, and B will accept in round 3. In either case, B will win one unit at price P_1 and A will one unit at no more than P_2.▌

Though Example 1 is somewhat trivial given the degenerate distributions, it shows that if the values are not too far apart, bidders coordinate a division of the units early in the auction. In the next example, we allow for incomplete information.

Example 2 (Nondegenerate Distribution): Suppose V_A is drawn from some distribution with support $[L_A, H_A]$ and V_B is drawn from some distribution with support $[L_B, H_B]$. Further suppose that (L_A, H_A, L_B, H_B) are such that for any $V_A \in [L_A, H_A]$ and any $V_B \in [L_B, H_B]$, conditions (*i*) and (*ii*) in Example 1 hold. Then in any Perfect Bayesian Equilibrium where bidders use weakly undominated strategies, each bidder wins one unit, the auction ends before reaching round 4, and revenues are at most $P_1 + P_2$.

Proof: For any beliefs that bidders might form, they know that conditions (*i*) and (*ii*) hold. A subgame perfection argument similar to that in the proof of Example 1 establishes that the auction ends as stated. ▌

Though our examples were very specialized, they show that it is the players using backward induction that leads to low revenue outcomes in the ascending auction. Though these examples require special asymmetries on the bidders' distributions this may be an artifact of the discreteness of the units for sale. If the quantity for sale is sufficiently finely divided then bidders may be able to "buy off" low valued bidders, letting the low valued bidders win a small portion of the quantity in return for keeping the prices low.

7 Conclusion

Multi-unit ascending auctions may be viewed as a negotiation between bidders on how to divide the available quantity. In an ascending auction with only two bidders who have complete and perfect information, we have shown that in the unique equilibrium bidders negotiate very rapidly. The first bidder proposes a split in the first round, and the second bidder accepts in the second round. A pressing issue for us is whether this result generalizes to more than two bidders. Also at issue is whether the uniqueness result extends when there are more than two bidders. In the alternating-offer bargaining game, there is uniqueness when just two bargainers are at the table, yet the uniqueness no longer holds when there are three bargainers (see pages 63-65 of Osborne and Rubinstein, 1990). The bargaining literature, however, typically does not assume that the terminal price is reached in bounded time. Nor does the bargaining literature typically assume that once a player proposes an offer, her future offers are constrained. Yet, such limitations are sensible in auctions. We conjecture that, in our framework, the uniqueness result does generalize to more than two bidders.

It follows from our paper that the sealed-bid uniform-price auction can do better than the ascending uniform-price auction. Suppose that the auctioneer imposes no quantity restrictions on the bidders. In the sealed-bid uniform-price auction, the auctioneer takes the bidders' downward-sloping bid functions that map quantities into prices, aggregates these functions, finds the price that clears the market (the price of the highest rejected bid), charges this price for all quantities awarded, and awards quantity corresponding to the bids above this price.[7] To make this auction consistent with the ascending auction that we have

[7] For the precise rules of a sealed-bid uniform-price auction see Ausubel and Cramton (1996).

presented, suppose that bid functions must be step-functions with steps at P_0, P_1, … Then there exists an efficient Nash equilibrium where each bidder bids for the entire quantity at the highest allowable bid below her value. In this equilibrium, the high valued-bidder will win the entire quantity and pay the highest bid increment below the low-valued bidder's value. This gives the efficient outcome, and raises more revenue than the ascending auction if there are two or more bid increments below the low-valued bidder's value. There may be other equilibria in the sealed-bid auction. One might be tempted to say that the set of Nash equilibrium outcomes in the sealed-bid auction are contained in the set of subgame perfect equilibrium outcomes since bidders might use complex punishment strategies to enforce different outcomes. Our paper, however, shows that this in not the case, since there exists the efficient outcome in the sealed-bid auction, but the efficient outcome does not obtain in the ascending auction.

Much research remains. First, given the severe prediction made by the ascending auction paradox, it is important to know the class of dynamic auctions to which the paradox extends. Clearly it does not extend to the alternative ascending auction proposed by Ausubel (1997), as the price is not uniform. It may extend to the FCC auction format, as intertemporal arbitrage tends to give the outcomes a uniform-price character. Though we modeled the auction format in our paper to resemble the FCC auctions, a key distinction is that in the FCC auctions, the licenses for sale are discrete and labeled. This means that prices can rise by different amounts on differently labeled licenses. This difference could make the backward induction argument much more complicated, though we believe the reasoning still applies. A second issue our research raises is how robust the ascending auction paradox is to incomplete information. We have shown with a stylized example that with some incomplete information, bidders still can use backward induction to justify settling the auction immediately at low prices. If in more general settings of incomplete information, it happens that information on bidders' signals is thrown out in attempt to settle the auction before the price rises, then some of the proposed benefits of an ascending auction will not materialize.

Appendix 1: Proof of Theorem 1

For this proof, define $q_{1,0} = q_{1,-1}$ as bidder 1's quantity restriction. By Definition 3, only offers $q_{B,S}$ by B in round S that satisfy: $[1 - q_{B,S}][V_A - P_{S-1}] \geq \bar{q}_{A,S+1}[V_A - P_{S+1}]$, or equivalently, $q_{B,S} \leq 1 - \delta_{A,S+1}\bar{q}_{A,S+1}$ are acceptable, where A's best acceptable offer in round S+1 is $\bar{q}_{A,S+1} = q_{A,0}$, her largest feasible offer. Thus, B's best acceptable offer in round S is $\bar{q}_{B,S} = \min\{q_{B,0}, 1 - \delta_{A,S+1}\bar{q}_{A,S+1}\}$. Any $q_{B,S} < \bar{q}_{B,S}$ is strictly acceptable, and must be accepted by A in round S+1 with probability one in any SPE. Any $q_{B,S} > \bar{q}_{B,S}$ must be rejected by A in round S+1 with probability one, since A can increase her payoff by counteroffering $q_{A,0}$ (or something very near $q_{A,0}$), which B will accept with probability one by Lemma 2. In round S, by rejecting an offer, B cannot obtain a higher payoff than $\bar{q}_{B,S}[V_B - P_S]$, but B can get arbitrarily close to this by counteroffering close to $\bar{q}_{B,S}$. In round S, B will not make an unacceptable offer since $\bar{q}_{B,S} > 1 - \bar{q}_{A,S+1} = 1 - q_{A,0}$ Bidder B will accept all strictly acceptable offers in round S, and will reject all unacceptable offers in round S, in favor of counteroffering $\bar{q}_{B,S}$. This counteroffer must be accepted with probability one in any SPE, since otherwise B would want to offer "just below" $\bar{q}_{B,S}$ so that no best response would exist for B in round S.

Assume that for rounds $s \in \{t+1,\ldots, S\}$ that: (*) For any subgame which begins in round s, in any SPE bidder *i* who moves in round s:

- does not make an unacceptable offer
- accepts with probability one any strictly acceptable offer
- rejects with probability one any unacceptable offer, and makes her best acceptable counteroffer, which is $\bar{q}_{i,s} = \min\{q_{i,0}, 1 - \delta_{s+1}\bar{q}_{j,s+1}\}$.

We complete the proof by showing (*) extends back to round t. It is simple to show $\bar{q}_{i,s} = \min\{q_{i,0}, 1 - \delta_{s+1}\bar{q}_{j,s+1}\}$. Bidder *i* does not make an unacceptable offer since $\bar{q}_{i,s} > 1 - \bar{q}_{j,s+1}$. That bidder *i* accepts with probability one strictly acceptable offers and rejects with probability one unacceptable offers follows from a parallel argument to that for round s = S above. This establishes that (*) extends back to round t. The theorem then follows noting that bidder 1's best acceptable offer must be accepted with probability one, otherwise bidder 1 wishes to offer "just below" her best acceptable offer so that no best response would exist.▮

Appendix 2: Proof of Theorem 2

The proof relies on a series of claims, each one proven in turn. The idea is to show that a bidder never wants to make an unacceptable offer. To do this, we need to characterize best acceptable offers.

Claim 1: For any subgame which begins in round S+1, in any SPE bidder A:

- accepts with probability one any strictly acceptable offer
- rejects with probability one any unacceptable offer, and makes her best acceptable counteroffer, $\overline{q}_{A,S+1} = q_{A,S-1}$.

Proof: By Definition 3, offers $q_{B,S}$ by B in round S are strictly acceptable only if: $[1 - q_{B,S}][V_A - P_{S-1}] > q_{A,S-1}[V_A - P_{S+1}]$, noting that $\overline{q}_{A,S+1} = q_{A,S-1}$, the supremum of feasible offers. Such offers must be accepted with probability one by A since rejecting gives a lower payoff. Offers $q_{B,S}$ by B in round S are unacceptable only if: $[1 - q_{B,S}][V_A - P_{S-1}] < q_{A,S-1}[V_A - P_{S+1}]$. Such offers must be rejected with probability one since A can win $q_{A,S-1}$ (or just below $q_{A,S-1}$) at price P_{S+1} since by Lemma 2, B accepts all $q_{A,S+1} < 1$ with probability one. Further in any SPE, if an unacceptable $q_{B,S}$ was offered in round S, then in round S+2 B must accept with probability one even if $q_{A,S+1} = 1$. Otherwise A would want to offer "just below" 1 in round S+1 so that no best response would exist for A.▌

Claim 2: For any subgame which begins in round S, in any SPE bidder B:

- does not make an unacceptable offer
- accepts with probability one any strictly acceptable offer
- rejects with probability one any unacceptable offer, and makes her best acceptable counteroffer, $\overline{q}_{B,S} = \min\{q_{B,S-2}, 1 - \delta_{A,S+1}q_{A,S-1}\}$.

Proof: The acceptability condition for a feasible offer $q_{B,S}$ is $[1 - q_{B,S}][V_A - P_{S-1}] \geq q_{A,S-1}[V_A - P_{S+1}]$, or equivalently, $q_{B,S} \leq 1 - \delta_{A,S+1}q_{A,S-1}$. Therefore B's best acceptable offer is $\overline{q}_{B,S} = \min\{q_{B,S-2}, 1 - \delta_{A,S+1}q_{A,S-1}\}$. Any offers $q_{B,S} < \overline{q}_{B,S}$ are strictly acceptable, so that by Claim 1, A will accept $q_{B,S}$ in round S+1. Bidder B does not make unacceptable counteroffers since $\overline{q}_{B,S} > 1 - q_{A,S-1}$. The remainder of the proof is parallel to the proof of Claim 1.▌

Claim 3 (Stand-Pat Claim): If bidder *i* makes an acceptable offer $q_{i,t}$ in round $t \leq S - 1$, and if bidder *j* makes an acceptable offer $q_{j,t+1}$ in round t+1, then $q_{i,t}$ is bidder *i*'s best acceptable offer in round t+2.

Proof: For $q_{i,t}$ to be acceptable in round t, it must satisfy: $[1 - q_{i,t}][V_j - P_{t-1}] \geq \bar{q}_{j,t+1}[V_j - P_{t+1}]$ or, rewritten, $q_{i,t} \leq 1 - \delta_{j,t+1}\bar{q}_{j,t+1}$, where $\bar{q}_{j,t+1}$ is the best acceptable offer *j* can make in t+1. Let j offer $q_{j,t+1} \leq \bar{q}_{j,t+1}$ in round t+1. Then any $q_{i,t+2}$ made in t+2 is strictly acceptable if it satisfies: $[1 - q_{i,t+2}][V_j - P_{t+1}] > q_{j,t+1}[V_j - P_{t+3}]$ or, rewritten, $q_{i,t} < 1 - \delta_{j,t+3}q_{j,t+1}$. But because discount factors are strictly decreasing, $q_{i,t} \leq 1 - \delta_{j,t+1}\bar{q}_{j,t+1} < 1 - \delta_{j,t+3}\bar{q}_{j,t+1} \leq 1 - \delta_{j,t+3}q_{j,t+1}$ and so $q_{i,t}$ is strictly acceptable in round t+2. Thus, $q_{i,t}$ is bidder *i*'s best acceptable offer in round t+2. ▌

Claim 4: If $q_{i,t}$ is acceptable in round $t \leq S - 1$, then bidder *j*'s best acceptable offer in round t + 1 is $\bar{q}_{j,t+1} = \min\{q_{j,t-1}, 1 - \delta_{i,t+2}q_{i,t}\}$.

Proof: Using the Stand-Pat Claim, the acceptability condition for $q_{j,t+1}$ becomes: $[1 - q_{j,t+1}][V_i - P_t] \geq q_{i,t}[V_i - P_{t+2}]$, or equivalently, $q_{j,t+1} \leq 1 - \delta_{i,t+2}q_{i,t}$. Imposing the eligibility constraint then gives $\bar{q}_{j,t+1}$. ▌

Claim 5: In round $t \leq S - 1$, a feasible offer $q_{i,t}$ is acceptable if and only if it satisfies

$$q_{i,t} \leq \max\left\{1-\delta_{j,t+1}q_{j,t-1}, \frac{1-\delta_{j,t+1}}{1-\delta_{j,t+1}\delta_{i,t+2}}\right\}. \qquad (**)$$

Proof: (Necessity) If $q_{i,t}$ is acceptable, then by Claim 4, $[1 - q_{i,t}] \geq \delta_{j,t+1}\times \min\{q_{j,t-1}, 1 - \delta_{i,t+2}q_{i,t}\}$; this is equivalent to (**). (Sufficiency) Likewise, $\bar{q}_{j,t+1} \leq \max\left\{1-\delta_{i,t+2}q_{i,t}, \frac{1-\delta_{i,t+2}}{1-\delta_{i,t+2}\delta_{j,t+3}}\right\}$ for $t < S - 1$ (and also for $t = S - 1$ by Claim 2). Therefore, a feasible $q_{i,t}$ is acceptable if it satisfies: $q_{i,t} \leq 1 - \delta_{j,t+1}\max\left\{1-\delta_{i,t+2}q_{i,t}, \frac{1-\delta_{i,t+2}}{1-\delta_{i,t+2}\delta_{j,t+3}}\right\}$, or rewritten, $q_{i,t} \leq \min\left\{1-\delta_{j,t+1}\frac{1-\delta_{i,t+2}}{1-\delta_{i,t+2}\delta_{j,t+3}}, \frac{1-\delta_{j,t+1}}{1-\delta_{j,t+1}\delta_{i,t+2}}\right\} = \frac{1-\delta_{j,t+1}}{1-\delta_{j,t+1}\delta_{i,t+2}}$, noting $\delta_{j,t+3} < \delta_{j,t+1}$ (Lemma 3). The claim follows noting that any feasible offer satisfying $q_{i,t} \leq 1 - \delta_{j,t+1}q_{j,t-1}$ must be acceptable since $\bar{q}_{j,t+1} \leq q_{j,t-1}$. ▌

Claim 6: In round $t \leq S - 1$, bidder *i*'s best acceptable offer is

$$\bar{q}_{i,t} = \min\left\{q_{i,t-2}, \max\left\{1-\delta_{j,t+1}q_{j,t-1}, \frac{1-\delta_{j,t+1}}{1-\delta_{j,t+1}\delta_{i,t+2}}\right\}\right\}$$

and all lower offers are strictly acceptable.

Proof: The expression for $\bar{q}_{i,t}$ follows immediately from Claim 5. Smaller offers are strictly acceptable follows from substituting "<" for "≤" in the acceptability condition in Claim 5's sufficiency proof. ▌

Remark: By Claim 6, a bidder's best acceptable offer depends only on the current eligibilities (each bidder's most recent offer), and no prior part of the history. Thus, we refer to $\bar{q}_{i,t}(h_t)$ as $\bar{q}_{i,t}(q_{i,t-2}, q_{j,t-1})$.

Claim 7: For any subgame which begins in round S – 1, in any SPE bidder A:

- does not make an unacceptable offer
- accepts with probability one any strictly acceptable offer
- rejects with probability one any unacceptable offer, and makes her best acceptable counteroffer.

Proof: By Claim 2, after A makes an unacceptable offer q^U in round S – 1, B will counteroffer $\bar{q}_{B,S} = \min\{q_{B,S-2}, 1 - \delta_{A,S+1}q^U\}$. If $\bar{q}_{B,S} = q_{B,S-2}$ then A could do better by accepting $q_{B,S-2}$ in round S – 1. Therefore, we need only consider unacceptable offers q^U such that $\bar{q}_{B,S} = 1 - \delta_{A,S+1}q^U$. It follows that by accepting in round S+1, A obtains the quantity $\delta_{A,S+1}q^U$ at price P_{S-1}.

Any feasible offer less than $\dfrac{1-\delta_{B,S}}{1-\delta_{B,S}\delta_{A,S+1}}$ is strictly acceptable by Claim 6 and will be accepted by B in round S with probability one by Claim 2. To show that A does not make an unacceptable offer in round S – 1, it suffices to show $\dfrac{1-\delta_{B,S}}{1-\delta_{B,S}\delta_{A,S+1}} > \delta_{A,S+1}$, or rearranging, $1 - \delta_{A,S+1} - \delta_{B,S}[1 - (\delta_{A,S+1})^2] > 0$. This holds if and only if $\delta_{B,S} < \dfrac{1}{1+\delta_{A,S+1}}$, which holds for all $\delta_{A,S+1} < 1$ if and only if $\delta_{B,S} < ½$. With constant bid increments, $\delta_{B,S} = \dfrac{V_B - P_S}{V_B - P_{S-2}} < \dfrac{P_{S+2} - P_S}{P_{S+2} - P_{S-2}} = \dfrac{2\Delta}{4\Delta} = ½$. Thus, A will not make an unacceptable offer in round S – 1. The remainder of the proof is parallel to the proof of Claim 1.▌

Claim 8: Assume that for rounds s ∈ {t+1,…, S – 1} that: (#) For any subgame which begins in round s, in any SPE bidder *i* who moves in round s:

- does not make an unacceptable offer
- accepts with probability one any strictly acceptable offer
- rejects with probability one any unacceptable offer, and makes her best acceptable counteroffer.

Then in round s = t the induction hypothesis (#) holds.

Proof: First, notice that rather than make any unacceptable offer in round t such that *j*'s best acceptable offer in round t+1 is $q_{j,t-1}$, *i* can do strictly better by accepting $q_{j,t-1}$ in round t at a lower price. Therefore we need only consider unacceptable offers q^U by *i* such that *j*'s best acceptable counteroffer is strictly less than $q_{j,t-1}$.[8] Given that bidder *j* will respond to q^U with $\overline{q}_{j,t+1} = \max\left\{1-\delta_{i,t+2}q^U, \frac{1-\delta_{i,t+2}}{1-\delta_{i,t+2}\delta_{j,t+3}}\right\}$, by accepting in round t+2 the quantity *i* will obtain at price P_t is $[1-\overline{q}_{j,t+1}] \leq \left[1-\frac{1-\delta_{i,t+2}}{1-\delta_{i,t+2}\delta_{j,t+3}}\right] = \left[\frac{\delta_{i,t+2}(1-\delta_{j,t+3})}{1-\delta_{i,t+2}\delta_{j,t+3}}\right]$. Alternatively, any of *i*'s feasible round t offers that are less than $\frac{1-\delta_{j,t+1}}{1-\delta_{j,t+1}\delta_{i,t+2}}$ are strictly acceptable. By the induction hypothesis (#) of this claim, bidder *j* will accept such offers with probability one in round t + 1.

To show that bidder *i* will not make an unacceptable offer, it suffices to show that $\frac{1-\delta_{j,t+1}}{1-\delta_{j,t+1}\delta_{i,t+2}} > \frac{\delta_{i,t+2}(1-\delta_{j,t+3})}{1-\delta_{i,t+2}\delta_{j,t+3}}$, or rearranging, $[1-\delta_{i,t+2}][1-\delta_{j,t+1}-\delta_{i,t+2}(\delta_{j,t+1}-\delta_{j,t+1}\delta_{j,t+3})] > 0$. Noting that $1-\delta_{i,t+2} > 0$ and substituting in the formulae for $\delta_{j,t+1}$ and $\delta_{j,t+3}$, it suffices to show that $\frac{[P_{t+1}-P_{t-1}]-\delta_{i,t+2}[P_{t+3}-P_{t+1}]}{V_j - P_{t-1}} > 0$, which holds for constant bid increments. This establishes that in round t, *i* will not make unacceptable offers.

The remainder of the proof is parallel to the proof of Claim 1.▌

Proof of Theorem 2: By Claim 8, bidder 1 will not make an unacceptable offer in round 1. The unique SPE outcome is for bidder 1 to make her best acceptable offer, which in round 2, bidder 2 must accept with probability one, since otherwise bidder 1 would wish to offer "just below" her best acceptable offer so that no best response would exist. If the quantity restrictions are sufficiently large (as stated in the Theorem), then by Claim 6, bidder 1's round 1 offer is $\frac{1-\delta_{2,2}}{1-\delta_{1,3}\delta_{2,2}}$.▌

[8] But of course in round 1, we need also consider unacceptable offers by 1 such that bidder 2's best acceptable offer in round 2 is $q_{2,0}$, since by the rules of the game, bidder 1 cannot accept in round 1. However, any unacceptable offer 1 makes in round 1 such that 2 responds with $q_{2,0}$ is strictly dominated by the offer $1-\delta_{2,2}q_{2,0}$.

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