An Efficient Ascending-Bid Auction

for Multiple Objects

by

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This paper is dedicated to William Vickrey (1914−1996).

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Abstract

In multiple-object environments where individual bidders may demand more than one object, standard methods of auction generally result in allocative inefficiency. This paper proposes a new ascending-bid method for auctioning multiple identical objects, such as Treasury bills or communications spectrum. The auctioneer announces a current price, bidders report back the quantity demanded at that price, and the auctioneer raises the price. Objects are awarded to bidders at the current price whenever they are "clinched," and the process continues until the market clears. With pure private values, the proposed (dynamic) auction yields the same outcome as the (sealed-bid) Vickrey auction, but may be simpler for bidders to understand and has the advantage of assuring the privacy of the upper portions of bidders' demand curves. With interdependent values, the proposed auction may still yield efficiency, whereas the Vickrey auction fails due to a problem which could be described as the "Champion's Plague."

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The auctions literature has provided us with two fundamental prescriptions guiding effective auction design. First, an auction should be structured so that — conditional on winning — the price paid by a player is as independent as possible of that player's own bids (William Vickrey, 1961). Ideally, the winner's price should depend solely on opposing participants' bids — as in the standard sealed-bid, second-price auction — so that each participant has full incentive to truthfully reveal her value for the good. Second, an auction should be structured in an open fashion which maximizes the information which is available to each participant at the time she places her bids (Paul R. Milgrom and Robert J. Weber, 1982a). When there is a common-value component to valuation and when bidders' signals are affiliated, an open ascending-bid format may induce participants to bid more aggressively (on average) than in a sealed-bid format, since participants can infer greater information about their opponents' signals at the time they place their final bids.

In single-object environments, these dual prescriptions are often taken to imply the desirability of the English auction and to explain its prevalence (see, for example, the surveys of Milgrom, 1987, and R. Preston McAfee and John McMillan, 1987). However, in multiple-object environments, nobody has before combined these two broad insights and taken them to their logical conclusion. The current article does precisely that: I propose a new ascending-bid auction format for multiple objects which literally takes heed of the two overriding auction-design prescriptions.

Simplicity or transparency to bidders should be viewed as one important attribute and advantage of the proposed auction. While the single-object Vickrey auction is well known, the multi-unit auction proposed by Vickrey in the same 1961 article remains relatively obscure even among economists, and is hardly ever advocated for real-world use. One reason seems to be that many believe it is too complicated for practitioners to understand. For example, Barry J. Nalebuff and Jeremy I. Bulow (1993), in comments to the Federal Communications Commission (FCC) on behalf of Bell Atlantic, describe the Vickrey auction only to conclude (p. 29): "However, experience has shown that even economics Ph.D. students have trouble understanding the above description. ... The problem is that if people do not understand the payment rules of the auction then we do not have any confidence that the end result will be efficient." By contrast, I will claim that my ascending-bid auction design is simple enough to be understood by any aficionado of baseball pennant races or similar sports contests.

Indeed, the subtlety of the Vickrey auction has been a problem even in experimental auction studies involving merely a single object. John H. Kagel, Ronald M. Harstad and Dan Levin (1987) found that bidders with affiliated private values behaved closer to the dominant strategy in ascending-clock auctions than in sealed-bid second-price auctions. They conclude (p. 1300): "The structure of the English clock auctions makes it particularly clear to bidders that they don't want to bid above their private values. Once the clock price exceeds a bidder's value, it is clear that competing further to win necessarily involves losing money. ... This enhanced capacity of the English clock institution to produce observational learning distinguishes it most clearly, on a behavioral level, from the second-price institution." Even in contexts where the traditional theory finds no informational advantages to a dynamic auction over a static auction, there are still likely to be simplicity or transparency gains associated with the ascending-bid design in putting Vickrey's sweeping insights into practical use.

Privacy-preservation of the winning bidders' values is a second attribute and advantage of the new ascending-bid auction. Noting that English auctions are quite prevalent while sealed-bid second-price auctions are exceedingly rare in the real world, Michael H. Rothkopf, Thomas J. Teisberg, and Edward P. Kahn (1990) offer an explanation for the nonuse of the Vickrey auction: bidders will be reluctant to truthfully reveal their private values in an auction if either there may be cheating by the auctioneer or there will be subsequent auctions or negotiations in which their private information is relevant to the outcome (and can be used against them).[1] For example, suppose that the government auctions a spectrum license which the highest bidder values at $1 billion but the second-highest bidder values at only $100 million. In a second-price auction, the high bidder is supposed to win the license but pay only $100 million. There are at least three problems here. First, there is likely to be a public relations disaster, as the ensuing newspaper headlines read, "Billion-dollar communications license given away for a fraction of its value." Second, there may be a problem of cheating by the seller: after opening the submitted bids, the auctioneer may ask his friend, "Mind if I insert a bogus $997 million bid in your name? It won't cost you anything, but it will make me a lot of money." Third, widespread knowledge of the winner's billion-dollar bid may imperil her subsequent bargaining position with equipment suppliers. Such considerations favor ascending-bid auctions, since winning bidders need not reveal their entire demand curves, only the portion below the winning price. For example, after an English auction in the hypothetical situation, all that is revealed is that the second-highest bidder valued the license at $100 million, and that the high bidder valued the license at something over $100 million.

However, ***allocative efficiency***, more than anything else, is the decisive attribute and advantage afforded by my auction. For a class of environments where bidders' assessments of valuation depend on the information possessed by opposing bidders, the main theorem of this article demonstrates that the proposed (dynamic) auction yields efficient outcomes. By way of contrast, it is also shown that the (sealed-bid) Vickrey auction cannot generate efficiency. Recall the Winner's Curse: in single-object auctions, a bidder's expected value conditional on winning is less than her unconditional expected value. Extending this reasoning to multi-unit auctions yields an effect which I refer to as the "Champion's Plague": a bidder's conditional expected value is decreasing in the number of units she wins. Hence, bidders even in the Vickrey auction have an informational incentive to shade their bids on higher quantities relative to lower quantities, upsetting efficiency (which requires there to be no differential shading).

The starting point for understanding the design proposed herein is to think of multiple-object auctions as "share auctions" (Robert Wilson, 1979). Recall that the classic English auction for a single object can be sensibly collapsed down to a sealed-bid auction, in which participants simultaneously submit bids, the auction "price" is defined to be the second-highest bid, and the highest bidder is awarded the object for this price. Analogously, existing ascending-bid auction designs for multiple objects can be sensibly collapsed down to a share auction, in which participants simultaneously submit bids consisting of demand curves, the auction "price" is defined to be the price at which supply equals demand, and each

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[1] Richard Engelbrecht-Wiggans and Charles M. Kahn (1991) and Rothkopf and Harstad (1995) also provide models emphasizing the importance of protecting the privacy of winners' valuations.

participant is awarded the quantity which she demanded at this uniform, market-clearing price. For example, the simultaneous multiple-round auction whereby the FCC auctioned 10 nationwide pager licenses in July 1994 resulted in essentially identical licenses selling for virtually identical prices. Similarly, the open, ascending-bid auction proposed (but, to date, not implemented) for U.S. Treasury securities would utilize a uniform, market-clearing price (Joint Report on the Government Securities Market, 1992, pp. B23−B24).

While uniform-price share auctions may seem to be the appropriate multiple-object generalization of the second-price auction for a single object, they in fact create incentives for bidders to engage in "demand reduction." This inevitably gives rise to allocative inefficiency, and often diminishes seller revenues. In Lawrence M. Ausubel and Peter C. Cramton (1996), for virtually any multiple-object environment in which (i) bidders have tastes for consuming more than one object and (ii) there is any private-values component to bidder valuations, we prove that any Bayesian-Nash equilibrium of a uniform-price auction yields an inefficient outcome (i.e., objects are allocated to bidders other than those who value them the most) with positive probability.[2]

By way of contrast, the Vickrey auction is an effective design when bidders with pure private values have tastes for consuming more than one object. By the "Vickrey auction" for identical objects, I mean the following sealed-bid auction procedure, proposed in Vickrey (1961). First, bidders simultaneously and independently submit demand functions specifying the quantity of objects which they desire at each possible price. (Quantities may be restricted to be integers, in the case of M indivisible objects, or may be allowed to be continuous, in the case of perfectly-divisible objects.) Second, the auctioneer determines the price at which supply equals demand, and awards each bidder the quantity she demanded at this market-clearing price. Third, the price which each bidder pays for an object is determined to be the bid which she displaces, i.e., the price charged each bidder for each object is the opportunity cost of assigning the object to that bidder.

In the case of M indivisible objects, each bidder submits up to M bids. Submitted bids are ranked in descending order, and the objects are awarded to the bidders associated with the M highest bids. A bidder pays the amount of the highest rejected bid (other than her own) for her first object, the second-highest rejected bid (other than her own) for her second object, and so on, paying the kth highest rejected bid (other than her own) for her kth object. For perfectly-divisible objects, the outcome for bidder i in the Vickrey auction is depicted in Figure 1. In that figure, $q_i(p)$ denotes the demand function submitted by bidder i, $M - q_{-i}(p)$ denotes the residual supply after subtracting out the demands of all other bidders, p^* denotes the market-clearing price if bidder i participates in the auction, and p^*_{-i} denotes the market-clearing price which would have resulted in the absence of bidder i. The Vickrey auction awards a quantity of $q_i(p^*)$ to bidder i, and requires a payment denoted P_i, which is the area of the shaded region in Figure 1. Thus, each participants' payment (conditional upon winning a given quantity) is independent of her own bids, embodying the first prescription of auction design.

[2]There are essentially only two multiple-object environments which escape the inefficiency theorem of Ausubel and Cramton (1996). First, when each bidder can consume at most a single unit of the good, efficiency is readily attainable (Milgrom and Weber, 1982b). Second, in a purely-common-value environment, every allocation is equally efficient.

The questions under consideration may now be stated:

- *Can the analogy between the English auction and the second-price auction be completed for multiple objects: i.e., Does there exist an ascending-bid auction for multiple objects whose static representation is Vickrey's sealed-bid auction, when bidders have pure private values?*

- *To the extent that the analogy can be completed, what is the comparison between the sealed-bid auction and the analogous ascending-bid auction, when each bidder's value depends on other bidders' information?*

This article provides substantial answers to each of these two questions. A new ascending-bid auction is proposed for multiple identical objects and close substitutes, which satisfies the following analogy: *The second-price auction is to the English auction, as the multi-unit Vickrey auction is to the auction proposed herein.* Furthermore, in a setting with interdependent values which strictly generalizes the Milgrom and Weber (1982a) framework, the new (dynamic) auction format outperforms the (static) Vickrey auction on efficiency considerations.

While the focus of this article is on auction efficiency, other research indicates that this also bears a close relationship to revenue maximization. Indeed, in the symmetric constant-marginal-values demand structure examined in this article, Ausubel and Cramton's (1996) Theorem 6 demonstrates that if the seller is constrained to distribute all units of the good (i.e., constrained to set a reserve price of zero), then the seller's expected revenue is maximized by using the efficient allocation rule.[3] The intuition for this result is that a bidder's willingness-to-pay in an auction is related to the amount in surplus she can expect to attain in the ultimate allocation; the more in gains from trade which are attained, the higher that participants are willing to bid. With more general demand structures, while the goals of revenue maximization and efficiency are only imperfectly aligned, they should not be viewed as necessarily running counter to one another.

Moreover, while the focus of most auction theory research has been on situations where resale of objects is precluded, it should be observed that the formal introduction of resale into the model is likely to only strengthen the conclusions of this article. Compare two trading structures in an environment with independent private values. Let the first trading structure consist of an auction which yields efficient allocations (so that no post-auction resale market is necessary), and let the second trading structure consist of an inefficient auction followed by an efficient resale market. Since both of these trading structures ultimately result in the efficient allocation of the objects, the total realized surplus from the two trading structures is equal, i.e.,

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[3]If the seller is not required to distribute all units of the object, then for a symmetric constant-marginal-values demand structure, Theorem 6 of Ausubel and Cramton concludes that the seller maximizes revenue by efficiently allocating the objects, subject to the imposition of a reserve price. This is readily accomplished using the auction format proposed in this article, only with the auction clock starting its ascent from the desired reserve price.

$$\text{Bidder Payoffs}_1 + \text{Seller Revenues}_1 + \text{Transaction Costs of Resale Market}_1 = \\ = \text{Bidder Payoffs}_2 + \text{Seller Revenues}_2 + \text{Transaction Costs of Resale Market}_2.$$

However, by the reasoning of the Revenue Equivalence Theorem, each trading structure should yield the bidders the same interim payoffs, i.e., Bidder Payoffs$_1$ = Bidder Payoffs$_2$. Meanwhile, the presence of broker commissions, trading delays, and the like means that Transaction Costs of Resale Market$_2$ > 0, whereas the Transaction Costs of Resale Market$_1$ = 0 (given that resale is unnecessary). Hence, we conclude that Seller Revenues$_1$ > Seller Revenues$_2$. The seller revenues from a market structure with an efficient auction should exceed the seller revenues from a market structure with an inefficient auction.

The following articles constitute a less-than-exhaustive list of related research. Edward H. Clarke (1971) and Theodore Groves (1973) introduce dominant-strategy mechanisms for dissimilar objects which parallel Vickrey's auction for identical objects. Milgrom and Weber (1982b, pp. 4−5) introduce the standard ascending-bid auction when bidders have unit demands and there are multiple identical objects, and extend their (1982a) analysis of symmetric environments with affiliated information to this multi-unit context. Kevin A. McCabe, Stephen J. Rassenti, and Vernon L. Smith (1990, 1991) study the standard ascending-bid auction when bidders have unit demands and there are multiple identical objects, obtaining both theoretical and experimental results. Eric S. Maskin (1992) demonstrates that, for single-object auctions with asymmetric bidders and interdependent information, the English auction is more likely to yield efficiency than the second-price auction. Maskin and John G. Riley (1989) examine optimal auctions for multiple identical objects in an independent private values setting. Partha Dasgupta and Maskin (1997) define a sealed-bid auction designed to attain efficiency with multiple dissimilar objects. Alexander S. Kelso and Vincent P. Crawford (1982), Gabrielle Demange, David Gale, and Marilda Sotomayor (1986), Sushil Bikhchandani and John W. Mamer (1996), Bikhchandani (1996), and Faruk Gul and Ennio Stacchetti (1997a,b) study various auction procedures for multiple objects and their relationship with Walrasian prices and/or efficient allocations under complete information. Vijay Krishna and Motty Perry (1997) study the Vickrey auction in an independent private values setting. In Ausubel and Raymond J. Deneckere (1993), we perform a related analysis in a different context, constructing a dynamic procedure which replicates the efficient static mechanism for incomplete-information bargaining.

The current article is organized as follows. Section 1 informally presents the new ascending-bid auction via an illustrative example which any aficionado of pennant races ought to understand. Section 2 continues to informally describe many of the main themes of the paper by showing why other auction formats yield inefficient outcomes for the illustrative example. Section 3 provides a formal specification of the auction as a continuous-time game. Section 4 establishes the main results under pure private values. Section 5 treats a “general symmetric model” where bidders' signals are affiliated and their values depend on other bidders' signals. Section 6 demonstrates that the new ascending-bid auction outperforms the Vickrey auction in the general symmetric model. Section 7 works out the equilibrium bidding strategies for a symmetric example. Section 8 concludes. Proofs are relegated to Appendix A, and a modified Vickrey auction for bidders with interdependent values is described in Appendix B.

1 An Illustrative Complete-Information Example

I begin by illustrating my proposal for an ascending-bid, multiple-object auction with an example loosely patterned after the FCC's auction for nationwide pager licenses in July 1994.

EXAMPLE 1. Suppose that there are five identical licenses for auction.[4] Each bidder has taste for more than one license, but bidders are each limited to winning at most three licenses.[5] There are six bidders with values in the relevant range, and their marginal values are given as follows (where numbers are expressed in millions of dollars):

(1)

Bidder A:	$v_{A,1}$	=	123
	$v_{A,2}$	=	113
	$v_{A,3}$	=	103
Bidder B:	$v_{B,1}$	=	75
	$v_{B,2}$	=	5
	$v_{B,3}$	=	3
Bidder C:	$v_{C,1}$	=	125
	$v_{C,2}$	=	125
	$v_{C,3}$	=	49
Bidder D:	$v_{D,1}$	=	85
	$v_{D,2}$	=	65
	$v_{D,3}$	=	7
Bidder E:	$v_{E,1}$	=	45
	$v_{E,2}$	=	25
	$v_{E,3}$	=	5
Bidder F:	$v_{F,1}$	=	49
	$v_{F,2}$	=	9
	$v_{F,3}$	=	3

The above are marginal values for a first, second, and third license, respectively. For example, if Bidder A were to purchase two licenses at prices of 80 each, his total utility from the transaction would be computed by: $v_{A,1} + v_{A,2} - 80 - 80 = 123 + 113 - 160 = 76$. In this example, bidders are presumed to possess complete information about their rivals' valuations.

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[4]In actuality, the FCC's nationwide narrowband auction offered ten licenses, of three different types: five (essentially identical) 50/50 kHz paired licenses; three (essentially identical) 50/12.5 kHz paired licenses; and two (essentially identical) 50 kHz unpaired licenses. For an extraordinarily cogent discussion of the nationwide narrowband auction, see Cramton (1995).

[5]In actuality, the FCC limited bidders to acquiring three licenses, either through the auction or through resale. Observe that the total number of licenses is not an integer multiple of each bidder's limitation on purchases, so with incomplete information, the inefficiency result of Ausubel and Cramton (1996, Theorem 1) is applicable, even if the marginal values for the first, second, and third licenses are equal. However, to make the following discussion straightforward, I will be assuming below that bidders have complete information about their competitors' values.

1.1 The Auction Play-by-Play

The proposed new auction is a particular specification of rules for what might be referred to generically as an "ascending-clock auction."[6] The auctioneer starts the auction at a reserve price of p^0 and proceeds to increment a continuously-ascending price clock. At each price, bidders simultaneously indicate how many licenses they are willing to purchase at the current unit price. For example, each bidder might be provided four cards stating "0", "1", "2" and "3", respectively, and each bidder holds up the appropriate card at each price. Alternatively, bidders may be provided with levers or buttons permitting them to declare their quantity at each price, or may be equipped with electronic terminals which permit them to enter their demands. The quantity is required to be a nonincreasing function of price;[7] and, for the moment, we will assume that each bidder can fully observe the quantities demanded by each of her rivals. When a price, p^*, is reached such that aggregate demand no longer exceeds supply, the auction is deemed to have concluded, and each bidder i is then assigned the quantity $q_i(p^*)$ she demanded at the final price. However, as we shall soon see, a winning bidder is *not* necessarily charged a unit price of p^*.

We shall now trace the progress of bidding by the posited bidders with marginal values given in eqs. (1), under the hypothesis that players bid sincerely; later, the sincere-bidding hypothesis will be justified. The bidding can also be followed in Table 1.

Let us say that p^0 = \$10 million. Then the auction opens with the auctioneer announcing a price of \$10 million. Bidders A−F indicate demands of 3, 1, 3, 2, 2 and 1, respectively. The aggregate demand is 12, whereas the available supply is only 5, so the auction must proceed further. The auctioneer begins to continuously increment the clock. The next change in demands occurs when the price reaches \$25 million. As $v_{E,2} = 25$, Bidder E drops her quantity demanded from 2 to 1, reducing the aggregate demand to 11. Still, the auction must proceed further. Since no bidder possesses a marginal value between \$25 million and \$45 million, at any quantity, nothing changes until the price reaches \$45 million.

There, however, the action begins to get more exciting. Bidder E drops out of the auction completely at \$45 million (since $v_{E,1} = 45$), Bidder F drops out of the auction completely at \$49 million (since $v_{F,1} = 49$), and Bidder C reduces her quantity demanded from 3 to 2 at \$49 million (since $v_{C,3} = 49$). The aggregate demand is now reduced to 8. In the unlikely event that a television sportscaster were reporting on this auction, the sportscaster could now remark (in standard American sportswriting jargon):[8]

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[6]The economics literature has sometimes referred to auction formats where the auctioneer — as opposed to bidders — continuously raises prices as the "Japanese auction" (see, for example, Milgrom and Weber, 1982a, p. 1104). However, this terminology appears to be based on an erroneous reading of Cassady (1967); and such an auction format appears to bear no relation to Japan. Consequently, this author seeks to help erase the "Japanese auction" terminology from the lexicon, and to instead encourage the use of the "ascending-clock auction" terminology.

[7]In the actual nationwide narrowband auction, the activity rule was that each bidder could demand up to three licenses in any round, provided that she had demanded *at least one* in the previous round. (See, Cramton, 1995, p. 337.)

[8]Readers not familiar with the usage of "clinch" and "magic number" in this article (or seeking insight into the history of thought behind the proposed new auction design) are referred to Joseph Durso, "Mets Down Cards, 3−2, in 11th and Hope to Clinch Title Today; Cubs Lose," *The New York Times*, September 24, 1969, p. 38:

> The Mighty Mets — who have never clinched anything more grand than ninth place — clinched at least a tie for the Eastern Division championship of the National League last night when they defeated

> "*Bidder A now has a magic number of one. The moment that any other bidder reduces her demand by a single unit, all competition for Bidder A's first license will have been mathematically eliminated. Nonetheless, it is still possible that Bidder A is going to go home from this contest without a prize.*"

Nothing further changes until a price of $65 million is reached, and until then the fans remain in suspense.

The outcome begins to be determined with some finality when the price crosses $65 million. Observe that this prompts Bidder D to reduce her quantity demanded from 2 to 1, as $v_{D,2} = 65$, dropping the aggregate demand to 7. The sportscaster could now accurately exclaim:

> "*Bidder A has now mathematically guaranteed herself at least one license! The aggregate demand of all competitors other than Bidder A has dropped to 4, so no matter what any bidder does now, Bidder A goes home with at least one license. Bidder A has clinched winning a license!*"

It is important to observe, at this juncture, that nothing irrevocable has yet occurred insofar as Bidders B, C, or D winning licenses; no other units have yet been "clinched". However, Bidder C's "magic number" has dropped to one; any further reductions by any bidder besides C assures Bidder C at least one license.

The excitement builds when the price reaches $75 million. Observe that this prompts Bidder B to drop out of the auction, as $v_{B,1} = 75$, reducing the aggregate demand to just 6, for an auction offering 5 licenses. The sportscaster's commentary continues:

> "*Bidder C now has also mathematically assured herself a license, and Bidder A now has clinched herself a second! No matter what any bidder does now, Bidder C knows she is going home with at least one license, and Bidder A knows she is going home with at least two.*"

The auction ends when the price attains a level of $85 million. Bidder D now drops out of the auction, as $v_{D,1} = 85$, reducing the aggregate demand to just 5, thus equating demand with supply.

> the St. Louis Cardinals, 3−2, in 11 innings. . . .
>
> As a result, the Mets won their fourth straight game and the 17th in their last 21, and they stretched their lead over the Chicago Cubs to six games with six to play.
>
> They also reduced their "magic number" for winning the uncontested prize to one. That is, one more victory for the Mets or one more defeat for the Cubs — who lost to Montreal earlier in the day — will pop the corks in Shea Stadium.

The reader is also referred to Leonard Koppett, "Mets Win and Clinch Eastern Crown," *The New York Times*, September 25, 1969, p. 1:

> With a flourish worthy of the occasion, the New York Mets officially won their first championship last night by beating the St. Louis Cardinals, 6−0, in the presence of 54,928 paying spectators fully prepared for the ecstasy of the victory.
>
> By scoring their 96th triumph of the baseball season, the Mets clinched first place in the Eastern Division of the National League . . .
>
> As the players raced for the clubhouse for their own celebration, thousands of spectators leaped railings and quickly filled the field, roaring and chanting "We're No. 1."

The author of this article (then nine years old) was one of the 54,928 paying spectators at the September 24 game.

Bidder A wins three licenses and Bidder C wins two licenses, which is the efficient outcome (placing all licenses in the hands who value them the most).

1.2 Payments in the Alternative Ascending-Bid Auction

Thus far, the auction description has coincided with standard specifications of an ascending-bid, multiple-object auction. What remain to be specified, however, are the payments owed by each of the winners. If we impose uniform-price rules — as in the open ascending-bid Treasury auction proposed in the Joint Report on the Government Securities Market, or as implied by arbitrage in the FCC's simultaneous multiple-round auction — each winner would be obliged to pay \$85 million per license, following the bidding of Section 1.1. (However, be cautioned that the sincere-bidding strategies of Section 1.1 would *not* constitute a Nash equilibrium under uniform-price rules, as we will shortly see.)

The play-by-play exposition of Section 1.1 makes it trivial to describe the payment rule advocated in this paper. Indeed, the "clinching" metaphor is quite descriptive, and permits the definition of the proposed new auction format:

> ALTERNATIVE ASCENDING-BID AUCTION. The auctioneer operates a continuously-ascending clock. For each price, p, each bidder i simultaneously indicates the quantity, $q_i(p)$, she desires, where demands are required to be nonincreasing in price. When a price, p^*, is reached such that aggregate demand no longer exceeds supply, the auction is deemed to have concluded. Each bidder i is then assigned the quantity $q_i(p^*)$, and is charged the standing prices at which she "clinched" the respective objects.

(The notion of "clinching" will be rigorously defined in eq. (2), below.)

Thus, corresponding to the play-by-play of Section 1.1, Bidder A would win one license for a price of \$65 million, one license for \$75 million, and one license for \$85 million. Bidder C would win one license for \$75 million and one license for \$85 million. Indeed, whenever bidders have pure private values, we will see in Section 4 below that sincere bidding is always an equilibrium, yielding an efficient assignment of the objects. Furthermore, given bidders' values as specified in eqs. (1), it is possible to show that — following elimination of weakly-dominated strategies — this is the unique assignment of the objects associated with any subgame-perfect equilibrium of the alternative ascending-bid auction.[9]

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[9]To begin, observe that it is weakly dominant for any bidder i to stay in for at least one object at all prices below $v_{i,1}$, and to drop out completely when the price crosses $v_{i,1}$. Consequently, 45 constitutes a floor on price: indeed, since all six bidders have $v_{i,1} \geq 45$, no objects can be clinched, in equilibrium, at prices below 45. Next, observe that Bidder A can assure herself three objects clinched at prices no higher than 65, 75 and 85, guaranteeing her surplus of at least 114, merely by maintaining a demand of 3; and Bidder C can assure herself two objects clinched at prices no higher than 75 and 85, guaranteeing her surplus of at least 90, merely by maintaining a demand of 2. By contrast, if A were to settle for one object, her surplus would be bounded above by 80; and if B were to settle for one object, her surplus would be bounded above by 82. Hence, each of Bidders A and C will never drop below a demand of 2 in equilibrium. Moreover, since Bidder F will never drop out until a price of 49 and Bidder B will never drop out until a price of 75, the most surplus that Bidder A could earn by winning two objects is to clinch one object at 49 and clinch one object at 75, for surplus of only 112. We conclude that Bidder A will never drop below a demand of 3 in equilibrium, establishing the uniqueness of the equilibrium assignment of the objects.

2 Illustrative Example Continued: Other Auction Formats

2.1 The Standard Ascending-Bid Auction

Most previous attempts to define simultaneous dynamic auctions of multiple objects have exhibited the ***uniform-price property***: similar objects tend to sell for similar prices. For example, in contrast to the alternative ascending-bid auction, one could have defined its uniform-price counterpart:

> STANDARD ASCENDING-BID AUCTION. The auctioneer operates a continuously-ascending clock. For each price, p, each bidder i simultaneously indicates the quantity, $q_i(p)$, she desires, where demands are required to be nonincreasing in price. When a price, p^*, is reached such that aggregate demand no longer exceeds supply, the auction is deemed to have concluded. Each bidder i is then assigned the quantity $q_i(p^*)$, and each bidder i is charged a unit price of p^*.

Suppose that bidders' values are commonly known to be exactly as described in eqs. (1), but the standard ascending-bid auction is used to sell the objects. In this particular example, one can argue that: (1) "sincere bidding" by all bidders is no longer a Nash equilibrium; (2) any equilibrium following elimination of weakly-dominated strategies is inefficient (compared to the equilibrium of the alternative ascending-bid auction, which is efficient); and (3) any equilibrium following elimination of weakly-dominated strategies results in lower seller revenues (compared to the equilibrium of the auction design advocated here).[10]

Let us suppose that Bidders B−F are using the "sincere bidding" strategy, and let us determine the best response of Bidder A. Consider the juncture described in the play-by-play above, at the moment when price crosses $75 million. Recall that Bidders A−F indicate demands of 3, 0, 2, 1, 0 and 0, respectively; see Row 6 of Table 1. Moreover, it is a weakly-dominant strategy for Bidder D to maintain a quantity of 1 until price reaches 85, and to then reduce her quantity to 0. Bidder A recognizes that she has two options: she can continue to bid sincerely, resulting in the winning of 3 objects at a price of 85; or she can immediately reduce her demand to 2, thereby stopping the auction at a price of 75. Observe that the second option yields higher payoff than the first option: Bidder A's surplus from 3 objects at 85 equals $84 million; whereas her surplus from 2 objects at 75 equals $86 million.[11] Thus, sincere bidding is not an equilibrium; and, indeed, the price never rises above 75 in any equilibrium surviving elimination

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[10]Before proceeding, the reader should be forewarned that the above example was constructed partly as a propaganda point, and that it is easy to also construct complete-information examples in which each of these conclusions is reversed. However, I believe that it is appropriate to make the propaganda point here, since there exists an independent private values specification for which each of the three conclusions holds as a general proposition. (To be somewhat more precise, Ausubel and Cramton, 1996, prove that when bidders have constant marginal values which are independently distributed, the uniform price auction will generally lead bidders to reduce their demands below sincere levels, do not possess efficient Nash equilibria, and yield lower seller revenues than the Vickrey auction and the alternative ascending-bid auction.) But the exposition is much more straightforward under complete information than with private information. So I apologize to readers for perhaps making more of a complete-information example than is justified.

[11]Moreover, if Bidder C performs the analogous calculation, she finds that she prefers winning 2 objects at 85 over winning 1 object at 75. Since Bidder C knows that the auction must conclude by the time price reaches 85 (given Bidder D's dominant strategy), Bidder C will not prematurely reduce her demand at a price of 75, forcing the hand of Bidder A.

of weakly-dominated strategies. Consequently, Bidder D must win a unit, implying that all equilibria of the standard ascending-bid auction which survive elimination of weakly-dominated strategies are inefficient.[12]

This inefficient equilibrium outcome yields \$18 million less in surplus (the difference between Bidder A's marginal value for 3 and Bidder D's marginal value for 1) than the efficient allocation under the alternative auction design. The outcome is also suboptimal from the perspective of seller revenues: it yields \$375 million in revenue; whereas the alternative auction design, despite giving away one license at a bargain \$65 million, yielded \$385 million in revenue.

2.2 Other Dynamic Auction Formats

Another dynamic approach for auctioning multiple objects is the simultaneous multiple round auction used by the Federal Communications Commission to assign spectrum licenses. Rather than the bidders indicating quantities of objects, the bidders actually name prices on individual objects; the auction is not deemed to have concluded for any single object until the bidding stops on all the objects. In such an auction format, there is a fairly strong tendency toward arbitrage, so that similar objects sell for similar prices. Most strikingly, in the real-world Nationwide Narrowband Auction on which Example 1 was patterned, the five virtually-identical 50/50 kHz paired licenses each sold for exactly \$80,000,000; however, some other auctions have displayed some degree of price discrepancy among similar licenses. To the extent that the FCC's simultaneous multiple round auction is conducted with small bid increments, we should expect outcomes to display the uniform-price character and therefore we should expect essentially the same inefficient equilibrium as from the standard ascending-bid auction.

Yet another approach to dynamically auctioning identical objects is to sell them successively, one after another, by successive English auctions. This, for example, is how Sotheby's attempted to auction seven satellite transponders in November 1981 (see Milgrom and Weber, 1982b). Observe that there is now a tendency toward *intertemporal* arbitrage, which again lends the auction process a uniform-price character. Again, we will be able to argue for Example 1 that it is not an equilibrium in undominated strategies for the objects to be allocated efficiently. If this were an equilibrium outcome, the price for each object would necessarily be at least 85 — the marginal value of unsatisfied Bidder D. An efficient equilibrium thus would again give Bidder A surplus of at most 84. However, Bidder A can deviate and guarantee herself surplus of at least 86 by "throwing" the first three auctions, if necessary, to Bidder C (two objects) and Bidder D (one object). Literally, Bidder A uses a strategy of only bidding up to a price of 75 (+ε) until Bidders C and D have won their respective units. Then, with the high marginal values out of the way, Bidder A can assure herself the last two objects at prices of 75 each, since 75 is then the highest remaining unsatisfied marginal value. Indeed, it would seem that the unique equilibrium outcome

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[12]Moreover, it is straightforward to argue that a price of 75, accompanied by Bidders A and C each winning two licenses apiece and Bidder D winning one license, is the *unique* subgame-perfect equilibrium outcome following elimination of weakly-dominated strategies. Observe that 45 is a floor on price, since all six bidders have marginal values for their first unit of at least 45. Furthermore, it is a weakly-dominated strategy for Bidder B to drop out of the auction at any price below 75 or for Bidder D to drop out at any price below 85; meanwhile, Bidders A and C each prefer winning 2 objects at 75 to winning 1 object at 45, guaranteeing excess demand at all prices up to 75.

in undominated strategies is for Bidder A to win 2 objects, Bidder C to win 2 objects, and Bidder D to win 1 object, all at prices of essentially 75.

2.3 Sealed-Bid Auction Formats

Let us also consider three sealed-bid formats for simultaneously auctioning multiple identical objects.[13] In each, the bidders simultaneously and independently submit up to M bids. The auctioneer then ranks the tendered bids in descending order and awards the objects to the bidders who tendered the M highest bids. The three auction formats differ in their payment rules. They are:

PAY-YOUR-BID AUCTION. A winning bidder pays the amount of her winning bids.

UNIFORM-PRICE AUCTION. A winning bidder pays a unit price equaling the highest rejected bid.

VICKREY AUCTION. A winning bidder pays the highest rejected bid (other than her own) for her first object, the second highest rejected bid (other than her own) for her second object, etc.

In Example 1, the pay-your-bid and uniform-price auctions again have the property that all equilibria in undominated strategies are inefficient. For the uniform-price auction, this follows the same argument as before: an efficient equilibrium in undominated strategies need have a highest rejected bid of 85, corresponding to Bidder D; and a next highest bid of 75, corresponding to Bidder B. Bidder A easily calculates that she can improve her payoff by reducing her third bid below 75, so that she wins only two objects. For the pay-your-bid auction, we can apply almost identical reasoning: in an efficient equilibrium, the winning bids must all be at least 85; otherwise, unsatisfied Bidder D could profitably deviate. But Bidder A could substitute two bids of 75 (+ε) for her three bids of 85 (+ε), as usual increasing her payoff.[14]

By contrast, in the Vickrey auction, each winning bidder is charged a payment corresponding to the opportunity cost of providing her the number of goods she wins. Thus, there are no incentives for misrepresentation by bidders. The unique equilibrium in undominated strategies is efficient, and with complete information, it corresponds precisely to the sincere-bidding equilibrium of the alternative ascending-bid auction.

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[13]Other sealed-bid auction formats are also possible. For example, Bikhchandani (1996) considers separate first-price auctions held simultaneously for each of the individual objects, as well as separate second-price auctions held simultaneously for each of the individual objects.

[14]The pay-your-bid and uniform-price auctions also possess efficient equilibria — but only if we allow bidders to use weakly-dominated strategies. For Example 1, following Bikhchandani (1996), it is an efficient equilibrium of the pay-your-bid auction if Bidder A submits three (winning) bids of 85 (+ε), Bidder C submits two (winning) bids of 85 (+ε), and Bidder D submits three (losing) bids of 85. However, observe that this requires Bidder D (or other losing bidders) to place more than one bid of 85, despite the fact that Bidder D's marginal values for a second object — and all other bidders' marginal values — are well less than 85. If only one (losing) bid of 85 is tendered, Bidder A can profitably deviate by instead bidding only 75 (+ε), and settling for only two objects. Similar reasoning applies to the uniform-price auction.

3 Formal Description of the Auction

3.1 Continuous-Time Game

The auction is formally modeled as a continuous-time game. However, our modeling should take into account the possibility that bidder *i*'s strategy may be to reduce her quantity at a given time, while bidder *j*'s strategy may be to reduce his quantity at the soonest possible instant after bidder *i* reduces her quantity. Given that the game is continuous in time, our modeling ideally should allow "moves that occur consecutively but at the same moment in time" (Leo K. Simon and Maxwell B. Stinchcombe, 1989, p. 1181).

The game may be conceptualized by thinking of "time" as being represented by a pair, (t,s), where t is given by a continuous ascending clock and s is given by a discrete ascending counter. Times are ordered lexicographically: first in t; and second in s. Generally speaking, the clock time t (and the associated price $p(t)$) increments continuously, and each bidder is free to reduce her quantity at any clock time. However, if bidder i reduces her quantity at a given clock time t, bidder j is allowed to respond by making his own reduction at the same clock time t (but, nevertheless, after bidder i's move). This may be conceptualized by using the counter s: if bidder i reduces her quantity at (t,s), the next available time which follows is $(t,s+1)$. Each time that some bidder reduces her quantity, the counter increments instead of the clock; and when players have finished reducing their quantities at the current clock time, the clock restarts instead.

If price is a continuous and strictly-increasing function of time (e.g., $p(t) = p^0 + t$), we may generally suppress both the clock t and the implicit counter s from our notation. A ***history*** h of the auction game consists of a string of pairs: $h \equiv (p^0,q^0), \ldots, (p^L,q^L)$. Together with the current price on the clock, h fully summarizes the play of the game. Each p^ℓ denotes the price at the ℓth occasion on which one or more bidders strictly decreased her quantity, and q^ℓ denotes the vector of quantities demanded by bidders $1,\ldots,n$ beginning at that occasion. For every ℓ $(0 \le \ell < L)$, we require that $p^{\ell+1} \ge p^\ell$. If $p^{\ell+1} > p^\ell$, our interpretation is that all of the bidders demanded constant quantities of q^ℓ at the half-open interval of prices $[p^\ell,p^{\ell+1})$, and one or more bidders reduced her quantity at price $p^{\ell+1}$. If $p^{\ell+1} = p^\ell$, our interpretation is that this is a situation where consecutive moves were made at the same clock time (and price). Let C^ℓ denote the vector of quantities clinched (defined formally in Section 3.2, below) by bidders $1,\ldots,n$ given the history truncated at (p^ℓ,q^ℓ). Since the auction rules require bidders to weakly decrease their quantities as the clock ascends, and given that clinched units have already been won, we also require: $C^\ell \le q^{\ell+1} \le q^\ell$ (with the second inequality holding strictly for some bidder, so that $p^{\ell+1}$ in fact refers to a price at which some bidder reduced her quantity).

Let x_i denote the private signal received by bidder i, and let $\theta_i(h)$ denote the summary of the history which is made observable to bidder i. If bidders are provided with full information about the past play, then $\theta_i(h) = h$. However, the auction process might only make partial information available: for example, $\theta_i(h)$ might be the sum of the quantities demanded by all the bidders in history h, or $\theta_i(h)$ might be a zero-one indicator of whether the auction is still open (in each case, together with the current quantity demanded by bidder i in history h).

A ***pure strategy*** for bidder i is a pair of functions, $\beta_i(x_i,\theta_i(h))$ and $\gamma_i(x_i,\theta_i(h))$, of private signals and observable summaries of the history. To simplify notation, let us assume that $\theta_i(h) = h$. Then $\beta_i(x_i,h)$ provides the lowest price at which bidder i will decrease her demand, and $\gamma_i(x_i,h)$ provides the quantity to which she will decrease her demand, given that the history is h. Thus, $\beta_i : \mathbb{R}\times H \to \mathbb{R}_+$ and $\gamma_i : \mathbb{R}\times H \to \mathbb{N}$, where $\mathbb{R}$ denotes the set of all possible (real-valued) signals, H denotes the set of all possible histories, $\mathbb{R}_+$ denotes the set of (nonnegative, real-valued) prices, and $\mathbb{N}$ denotes the set of (nonnegative, integer-valued) quantities. To be consistent with the rules, $\gamma_i(x_i,h) < q_i(h)$, the quantity demanded under the current history h. Furthermore, if bidder i (impermissibly) attempts to reduce her quantity below the amount $C_i(h)$ she has already clinched, the bid which will be entered is $\gamma_i(x_i,h) = C_i(h)$. Whenever the auction reaches a point where the quantity $q_i(h)$ demanded by bidder i in history h equals the amount $C_i(h)$ she has already clinched, bidder i no longer has available any legal moves and so subsequently is no longer treated as an active player in the game.

Observe that the component $\beta_i(x_i,h)$ of strategy may be interpreted as providing the price p at which bidder i reduces her quantity, given history h and the hypothesis that no other bidder j first reduces his quantity at any price reached before p. (The strategy need not incorporate the possibility that some other bidder does reduce his quantity first, since then the history changes from h to h', and so bidder i would instead play according to the strategy $\beta_i(x_i,h')$.) It is straightforward to see how any n-tuple of strategies generates a unique history. Suppose that a history h^ℓ of length ℓ and strategies $\beta_i(x_i,h^\ell)$ $(i=1,\ldots,n)$ are given. Define $p^{\ell+1} = \min\{\beta_i(x_i,h^\ell) \mid i=1,\ldots,n\}$; define $I^{\ell+1} = \{i \mid \beta_i(x_i,h^\ell)=p^{\ell+1}\}$; and define $q_i^{\ell+1} = \gamma_i(x_i,h^\ell)$ if $i\in I^{\ell+1}$, and $q_i^{\ell+1} = q_i^\ell$ otherwise. These definitions extend the history of length ℓ to a unique history of length $\ell+1$.

We will now consider two possible specifications of the auction rules.

3.2 Basic Specification of the Auction Rules

For any ℓ, define $Q^\ell = \sum_i q_i^\ell$ to be the aggregate demand by all bidders. We will refer to the string $h \equiv \{(p^0,q^0), \ldots, (p^L,q^L)\}$ as a ***final history*** if $Q^L \le M$ but $Q^{L-1} > M$, since the auction then concludes at price p^L. We now define the allocation and payment associated with any final history, for the basic specification of the auction rules.

First, suppose that $Q^0 < M$. Then we will say that the auction has "failed": the available quantity was not fully subscribed at the reserve price. Next, suppose that $Q^0 \ge M$. Then the auction has succeeded in the sense of fully allocating the available quantity of units at no less than the reserve price. We can define the cumulative vector of quantities C^ℓ ***clinched*** at prices up to p^ℓ by:

(2) $$C_i^\ell = \max\{0, M - \sum_{j\neq i} q_j^\ell\}, \text{ for } \ell=0,\ldots,L-1 \text{ and } i=1,\ldots,n.$$

For $\ell = L$, we may need to modify Eq. (2) by introducing rationing, if $Q^L < M$. Various rationing rules are possible; for example, we may specify proportional rationing, and so the following amounts will be expected quantities assigned to the respective bidders:

(3) $$C_i^L = q_i^L + \left[\frac{q_i^{L-1} - q_i^L}{Q^{L-1} - Q^L}\right]\left[M - Q^L\right], \text{ for } i = 1, \ldots, n.$$

Finally, from Eqs. (2)−(3), which provide cumulative amounts C_i^ℓ clinched at prices up to p^ℓ, we may define individual amounts c_i^ℓ newly clinched at price p^ℓ by setting $c_i^0 = C_i^0$ and:

(4) $$c_i^\ell = C_i^\ell - C_i^{\ell-1} \text{ for } \ell = 1, \ldots, L \text{ and } i = 1, \ldots, n.$$

Given the quantities clinched, as defined in Eqs. (2)−(3)−(4), we may now easily define the auction outcome associated with any final history:

(5) $$\text{Allocation: } \quad q_i^* = C_i^L, \text{ for } i = 1, \ldots, n,$$

(6) $$\text{Payment: } P_i^* = \sum_{\ell=0}^{L} p^\ell c_i^\ell, \text{ for } i = 1, \ldots, n.$$

Finally, observe that the formal specification of the alternative ascending-bid auction of Sections 3.1 and 3.2 can be trivially modified to also provide a formal specification of the standard ascending-bid auction. In the foregoing development, replace Eq. (2) with:

(2′) $$C_i^\ell = 0, \text{ for } \ell = 0, \ldots, L-1 \text{ and } i = 1, \ldots, n.$$

That is, if all "clinching" before the final price is discarded, then the basic specification of the alternative ascending-bid auction is immediately transformed into a basic specification of the standard ascending-bid auction.

3.3 "Turning-Back-the-Clock" Specification of the Auction Rules

The basic auction rules, as specified in the previous Section 3.2, may be understood to have the auction clock continuously ascend and — whenever, at any price p^ℓ, a bidder reduces her quantity demanded — the clock pauses and then resumes ascending, restarting at the same price p^ℓ where it stopped. However, there is no conceptual or game-theoretic reason why the clock necessarily need restart at the same price as where it stopped. More generally, for any history $h \equiv (p^0, q^0), \ldots, (p^\ell, q^\ell)$, we could just as easily have specified that the clock resume its continuous ascent starting at any price $r(h)$, where $r(h)$ need not necessarily equal the price p^ℓ where it stopped. Furthermore, once the function $r(\bullet)$ is specified, we will see that it is still possible to recover the full price path from the history h (as in Eqs. (7)−(8), below).

It will be seen in Section 4, below, that there is no particular motivation to use the more general specification in a world of pure private values. However, in Section 5, where bidders' values depend nontrivially on each others' signals, we will see that there may be efficiency gains in specifying a restart function $r(\bullet)$ with the property that $r(h) < p^\ell$ whenever $q_j^{\ell-1} > q_j^\ell = 0$, i.e., whenever some bidder j reduces her quantity to zero at p^ℓ. (If this is not the case, we maintain $r(h) = p^\ell$.)

Generally, given any restart function $r(\bullet)$ and any final history $h \equiv (p^0,q^0), \ldots, (p^L,q^L)$, the progression of prices on the auction clock will be defined recursively, as follows. We will want t^ℓ to denote the time (in the underlying continuous-time game) at which price p^ℓ appears on the clock, and we will want $p(t)$ to denote the complete mapping from times to prices (but observe that $p(t)$ may now be discontinuous and so can no longer always reflect the prices at which the clock is restarted). For convenience, let us parameterize price as a function of time in such a way that $dp/dt = 1$ on all segments where price is continuously ascending. To begin, for $\ell = 0$, we define $t^0 = 0$ and, of course, $p(t^0) = p^0$. Now suppose that $t^0,\ldots,t^\ell$ and $p(\bullet)$ (for $t \in [t^0,t^\ell]$) have already been defined. We define $t^{\ell+1}$ inductively by:

$$t^{\ell+1} = t^\ell + \left[p^{\ell+1} - r(h^\ell) \right] , \tag{7}$$

where h^ℓ denotes final history h truncated at (p^ℓ,q^ℓ). The function $p(\bullet)$ is extended to the interval $(t^\ell,t^{\ell+1}]$ by:

$$p(t) = r(h^\ell) + \left[t - t^\ell \right], \text{ for } t \in (t^\ell, t^{\ell+1}] , \tag{8}$$

where it should be observed that Eq. (8) vacuously extends the definition of $p(\bullet)$ if $t^{\ell+1} = t^\ell$.

One simple example of a restart function with the desired "turning-back-the-clock" property is developed as follows. Given any price path $p(t)$ already specified for $t^0 \le t \le t^\ell$, we can define:

$$p^{\max}(t^\ell) = \max\left\{ p(t) \mid t \in [t^0, t^\ell] \right\}, \tag{9}$$

Using Eq. (9), we may easily define a simple restart function, $r(\bullet)$, for any truncated history h^ℓ, by:

$$r(h^\ell) = \begin{cases} Rp^{\max}(t^\ell) , & \text{if } q_j^{\ell-1} > q_j^\ell = 0 \text{ for some } j = 1, \ldots, n , \\ \\ p^\ell , & \text{otherwise} , \end{cases} \tag{10}$$

where R is a positive constant strictly less than one. The restart function specified by Eq. (10) has the following straightforward description. Generally speaking, the clock price ascends according to $p(t) = p^0 + t$, with pauses at each time that some bidder reduces her quantity demanded. However, at each time t^ℓ that some bidder drops completely out of the auction, the clock is turned back to a price of $Rp^{\max}(t^\ell)$, i.e., a proportion $R < 1$ of the highest price which has thus far been attained. The clock then resumes its usual ascent.

Apart from this periodic turning-back of the clock, the formal specification of the game is analogous to that of Sections 3.1 and 3.2, above. Obviously, there is no longer any requirement in histories that $p^{\ell+1} \ge p^\ell$. We will now say that an auction has failed if not all units are clinched at prices of at least p^0. A pure strategy may still be specified in terms of the lowest price at which a bidder will decrease her demand given the history, since price is still *piecewise* a continuous and strictly-increasing function of time. The clinching, allocation, and payment rules continue to be defined by Eqs. (2) to (6).

In the following sections, we proceed to study the equilibria of the specified auction games.

4 Pure Private Values

Assume that a seller offers M indivisible units of a homogeneous good to n bidders. Each bidder, i, obtains marginal utility of $V_{i,k}$ from her kth unit of the good, for $k = 1, \ldots, M$. Thus, if bidder i obtains q_i units of the good for a total payment of P_i, she obtains a payoff of:

$$\sum_{k=1}^{q_i} V_{i,k} - P_i \text{ , for } i = 1, \ldots, n \text{ and } q_i = 1, \ldots, M \text{ .}$$

All bidders are assumed to exhibit (weakly) diminishing marginal utility, i.e., $V_{i,k} \geq V_{i,k+1} \geq 0$, for all $i = 1, \ldots, n$ and $k = 1, \ldots, M-1$. The marginal utilities, $V_{i,k}$ ($i = 1, \ldots, n$ and $k = 1, \ldots, M$) are privately observed by the respective bidders, and are allowed to arise from an arbitrary joint distribution; thus, $(V_{i,1},\ldots,V_{i,M})$ and $(V_{j,1},\ldots,V_{j,M})$ may be statistically independent of each other for $i \neq j$, or they may be correlated in arbitrary ways. However, bidders have ***pure private values*** in the sense that bidder i's estimation of her own value is not affected by bidder j's estimation of his own value. (A symmetric model with interdependent values will be analyzed in Section 5.)

As we allowed in Section 3.1, a wide variety of specifications of bidders' information is possible. Again, let $h \equiv (p^0,q^0), \ldots, (p^L,q^L)$ denote the current history and let $\theta_i(h)$ denote the summary of the history which is made observable to bidder i. Also, let p denote the current price on the ascending clock. Bidder i is always assumed to be able to observe p and her own demand q_i^L. Three of the most sensible of possible informational rules will now be considered:

FULL BID INFORMATION: The summary of the history observable to bidder i is: $\theta_i(h) = h$, i.e., the complete history of all bids by all bidders.

AGGREGATE BID INFORMATION: The summary of the history observable to bidder i is: $\theta_i(h) = (p^0, \sum_j q_j^0), \ldots, (p^L, \sum_j q_j^L)$, i.e., the history of the aggregate demand of all bidders.

NO BID INFORMATION: The summary of the history observable to bidder i is: $\theta_i(h) = 1$, if $\sum_j q_j^L > M$; and 0, otherwise, i.e., whether the auction is still open.

The notion of "turning-back-the-clock" which we developed in Section 3.3 will not be necessary in the pure private values context. Let us also define what is meant by "sincere bidding" in the context of a dynamic auction:

SINCERE BIDDING: The use of the following strategy pair by bidder i after every history $h \equiv (p^0,q^0), \ldots, (p^L,q^L)$ will be referred to as ***sincere bidding***:

$$\beta_i(x_i,h) \;=\; \begin{cases} V_{i,q_i^L}\,, & \text{if } V_{i,q_i^L} \ge p^L\,, \\ p^L\,, & \text{otherwise,} \end{cases} \qquad \gamma_i(x_i,h) \;=\; \begin{cases} \max\{k \mid V_{i,k} > \beta_i(x_i,h)\}\,, & \text{if } V_{i,1} > \beta_i(x_i,h)\,, \\ 0\,, & \text{otherwise.} \end{cases}$$

The basic results for the case of pure private values are the following two theorems. The first theorem is most easily argued by observing that, with pure private values, the alternative ascending-bid auction with no bid information is outcome-equivalent to the Vickrey auction, for which truthtelling by all bidders is an efficient equilibrium in weakly-dominant strategies (Vickrey, 1961). The essence of the argument is that in either the Vickrey auction or the alternative ascending-bid auction with opponents' strategies fixed, the only effect of changing one's own bid strategy is to alter the quantity, q_i^*, of units that bidder i wins. However, conditional on the quantity of units won, the payment P_i^* of bidder i is invariant.

THEOREM 1. In the alternative ascending-bid auction with pure private values, (weakly) diminishing marginal utilities, and no bid information, sincere bidding by every bidder is an efficient equilibrium in weakly-dominant strategies.

Theorem 1 and all subsequent theorems are proved in Appendix A.

Once full or partial bid information is introduced, the alternative ascending-bid auction loses its equivalence with the Vickrey auction. In particular, the dominant-strategy property is lost. For example, suppose that for some bizarre reason, bidder j uses the strategy of maintaining $q_j = q_j^0$ so long as $q_i = q_i^0$, but dropping to $q_j = 0$ at the first available moment that $q_i < q_i^0$. Then it is possible that bidder i may strictly improve her payoff by decreasing her quantity to $q_i^0 - 1$ at a price p where her marginal utility for a q_i^0th unit still exceeds p. While the dominant-strategy property is lost, sincere bidding remains an equilibrium of the auction:

THEOREM 2. In the alternative ascending-bid auction with pure private values, (weakly) diminishing marginal utilities, and either full bid information or aggregate bid information, sincere bidding by every bidder constitutes an efficient equilibrium.

By contrast, Ausubel and Cramton (1996, Theorems 1 and 5) demonstrate that, in essentially any pure private values environment, the (sealed-bid) uniform-price auction does not possess any efficient equilibria. [The one exception to the Inefficiency Theorem in a world of private values is that, if each bidder has a constant marginal value up to a capacity of λ, and if $K \equiv M/\lambda$ is an integer, then the model is essentially the same as that of a K-object auction where bidders have unit demands, and so the uniform-price auction admits an efficient equilibrium.] The same reasoning easily extends to the standard ascending-bid auction. Thus, the Vickrey auction and the alternative ascending-bid auction have the substantial advantage — over their uniform-price counterparts — that they enable efficient outcomes in a pure private values environment. As emphasized in the Introduction, the alternative ascending-bid auction also offers advantages over the Vickrey auction: the auction may be simpler for participants to understand; and protects the privacy of high values.

5 Interdependent Values

In this section, we will depart from the world of pure private values, to consider a model in which there is a common-value component to bidders' values, i.e., any bidder's value depends both on her own signal and the other bidders' signals. Bidder strategy will then differ between a dynamic auction and the corresponding static auction, since a bidder may potentially learn or infer aspects of other players' information in the course of the game. The model to be analyzed is symmetric, in two respects. First, it is symmetric *between* bidders, in the sense that their signals are drawn from the same distributions and their values are given by the same functions of their own and others' signals. (However, different bidders will be permitted to have different capacities for the objects: an asymmetry which inevitably must be allowed, since even starting with symmetry, it occurs in subgames which need to be considered.) Second, the model is symmetric *within* bidders, in the sense that a bidder's value for her first, second, third, etc. object — up to her capacity — is equal.

However restrictive the model may be, the reader should observe that it is a strict generalization of the general symmetric model of Milgrom and Weber (1982a,b). That is, restrict the number M of objects to equal one, and this reduces precisely to the (1982a) model; allow $M > 1$ but restrict $\lambda_i = 1$ for $i = 1, \ldots, n$, and this reduces precisely to the (1982b) model.

5.1 The General Symmetric Model

Assume that a seller offers M indivisible units of a homogeneous good to n bidders. Each bidder, i, obtains constant marginal utility of V_i from each of up to λ_i units of the good, but zero marginal utility from any more than λ_i units. Thus, if bidder i obtains q_i units of the good for a total payment of P_i, she obtains payoff of $q_i V_i - P_i$, for $q_i = 0, \ldots, \lambda_i$ and $0 < \lambda_i \leq M$. We will refer to λ_i as the ***capacity*** of bidder i. Let the capacities be sufficiently large that there is competition for every unit of the good. The marginal values V_i $(i = 1, \ldots, n)$ are assumed to derive from affiliated signals. Let $X \equiv (X_1, \ldots, X_n)$ be a vector of n real-valued signals which are privately observed by the n respective bidders. Also let X_{-i} denote the $(n-1)$ signals observed by all agents except i, without the identities of the individual bidders indicated. Following Milgrom and Weber (1982a), it will be assumed that:

A.1 $V_i = u(X_i, X_{-i})$, where $u(\bullet)$ is the same nonnegative-valued function for each i $(i = 1, \ldots, n)$, $u(\bullet)$ is continuous in all its arguments, $u(\bullet)$ is strictly increasing in its first argument, and $u(\bullet)$ is nondecreasing in its remaining arguments.[15]

A.2 $E[V_i] < \infty$, for each i $(i = 1, \ldots, n)$.

A.3 The variables $(X_1, \ldots, X_n)$ are affiliated.

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[15] As noted by Milgrom and Weber (1982a, p. 1100), the "nondegeneracy assumption" that a bidder's expected value is strictly increasing in her own signal is unnecessary for the results to hold, but greatly simplifies the proofs.

A.4 The joint density, $f(\bullet,\ldots,\bullet)$, of $(X_1,\ldots,X_n)$ is symmetric in its arguments.

Loosely speaking, the affiliation assumption A.3 requires that the agents' signals, X_i, are nonnegatively correlated with one another. More precisely, let x and x' (each points in $\mathbb{R}^n$) be possible realizations of $(X_1,\ldots,X_n)$. Let $x \vee x'$ denote the componentwise maximum of x and x', and let $x \wedge x'$ denote the componentwise minimum. We say that $(X_1,\ldots,X_n)$ are ***affiliated*** if $f(x \vee x')\, f(x \wedge x') \geq f(x)\, f(x')$, for all x and x' (see Milgrom and Weber, 1982a, p. 1098). Let us also define:

> EFFICIENCY: An equilibrium is ***efficient*** if the objects are assigned to the bidders with the highest signals, for almost every realization of the signals $(X_1,\ldots,X_n)$. An equilibrium is ***allocatively-efficient*** if the objects are assigned to the bidders with the highest values, for almost every realization of the signals $(X_1,\ldots,X_n)$.

Clearly, it is straightforward to define an additional assumption under which efficiency and allocative efficiency coincide. Let X_{-ij} denote any $(n-2)$-tuple of signals received by all bidders except i and j. We may assume:

A.5 $u(X_i;X_j,X_{-ij}) > u(X_j;X_i,X_{-ij})$ whenever $X_i > X_j$.

Moreover, allocative efficiency is typically the concept which will be of economic interest. Nevertheless, all of the following results will not depend on assumption A.5; we add this as an assumption whenever we wish to be able to make a statement concerning allocative efficiency.

If desired, one could also add a vector $S \equiv (S_1,\ldots,S_m)$ of additional variables, some of which might be observed by the seller, which also influence the value of the objects to the bidders (as in Milgrom and Weber, 1982a). In the interest of notational brevity, this will not be done here.

5.2 An Efficient Equilibrium of the Alternative Ascending-Bid Auction

In this section, we utilize the "turning-back-the-clock" specification of the auction rules of Section 3.3, and we construct a simple and efficient equilibrium of the alternative ascending-bid auction.[16] Subsequent sections will compare the constructed equilibrium with those of other sealed-bid and ascending-bid auction forms.

The intuition behind the constructed equilibrium is rather straightforward. The equilibrium bidding threshold, $\beta(x,h)$, will be strictly increasing in the bidder's signal x, for each equilibrium history h. Suppose that the auction is "one bidder away" from each bidder clinching a positive number of objects. Then each bidder should bid up to her expected value for the objects, *conditional on the lowest of the other active bidders' signals equaling her own signal*. By doing so, she clinches objects precisely in those situations where her expected value exceeds the price for the clinched objects, and she fails to clinch

[16]The careful reader may wish to note that, if we had instead utilized the basic specification of the auction rules of Section 3.2, the equilibrium construction would have been much more complex. Moreover, it is unlikely that the resulting equilibrium would have been efficient, except when $\lambda_i = \lambda$ for all $i = 1,\ldots,n$.

objects precisely in those situations where her expected value is less than the price for clinching the objects.

Meanwhile, a bidder never need concern herself with the consequences of two bidders simultaneously decreasing their quantities. Within any given history h, simultaneous reduction is a zero-probability event, since the joint distribution of signals has no mass points and each bidding threshold $\beta_i(x,h)$ is strictly increasing in signal x. Between histories h^ℓ and $h^{\ell+1}$, bidders do reduce their bidding thresholds (i.e., $\beta_i(x,h^{\ell+1}) < \beta_i(x,h^\ell)$), and in general this could give rise to simultaneous reductions; however, by "turning back the clock" sufficiently far, the auction design can completely avoid this problem.

Finally, suppose that the auction is "two or more bidders away" from a given bidder i clinching a positive number of objects. Then bidder i does equally well with a wide array of strategies, since in equilibrium there is no danger of her clinching until some other bidder first withdraws — and she can withdraw immediately thereafter. One convenient available choice is to again have bidder i bid up to her expected value for the objects, conditional on the lowest of the other active bidders' signals equaling her own signal. By specifying strategies in this way, bidders demanding larger quantities, who may clinch at the next withdrawal, and bidders demanding smaller quantities, who cannot immediately clinch, use identical strategies, yielding efficiency even in asymmetric quantity situations.

In an auction with n initial bidders, define $N = \{1,\dots,n\}$, and for any bidder i ($i \in N$), define $N \setminus \{i\}$ = $\{j \in N \mid j \neq i\}$. For any j ($j = 1,\dots,n-1$), let Y_j^{-i} denote the jth-order statistic of the signals of the bidders $N \setminus \{i\}$, that is, the jth highest signal received by all the bidders excluding bidder i. Using the symmetry assumption A.4, the distribution of Y_j^{-i} is independent of i, and so the superscript "$-i$" will henceforth be suppressed from Y_j^{-i}.

Define a bidder i to be ***active*** after history h if and only if $q_i(h) > C_i(h)$. Furthermore, define $J(h) = |\{i \in N \mid q_i(h) > C_i(h)\}|$ to be the cardinality of the set of active bidders after history h. Then $n - J(h)$ bidders have dropped out at history h. Let bidder i be one of the remaining active bidders and suppose that the bidders who have dropped out correspond to the order statistics $Y_{J(h)},\dots,Y_{n-1}$. Let us define:

$$(11) \qquad v_j(x,y \mid y_j,\dots,y_{n-1}) = E[V_i \mid X_i = x, Y_{j-1} = y, Y_j = y_j, \dots, Y_{n-1} = y_{n-1}], \text{ for } j = 2,\dots,n.$$

We now define the equilibrium bidding threshold for any active bidder i to be her expected value for the objects, conditional on the lowest of the other active bidders' signals equaling her own signal x (and on the inferred realizations of $Y_{J(h)},\dots,Y_{n-1}$). Algebraically, this is expressed by:

$$(12) \qquad \beta(x,h) = v_{J(h)}(x,x \mid y_{J(h)},\dots,y_{n-1}) \quad \text{and} \quad \gamma_i(x,h) = C_i(h),$$

where $J(h)$ is the number of active bidders, and $y_{J(h)},\dots,y_{n-1}$ are the realizations of $Y_{J(h)},\dots,Y_{n-1}$, respectively, inferred from the history h and the equilibrium strategies. Further observe that the bidding threshold $\beta(x,h)$ of Eq. (12) is independent of i, and that if all bidders use $\beta(x,h)$, then the equilibrium inferences $y_{J(h)},\dots,y_{n-1}$ are given implicitly by the solutions in y_j to:

(13) $$\beta(y_j, h(j+1)) = p_{j+1}\,, \text{ for } j = J(h), \ldots, n-1\,,$$

where $h(j+1)$ denotes the extant history when the $(j+1)$st active bidder dropped out of the auction, p_{j+1} denotes the price at which the $(j+1)$st active bidder dropped out, and the shift from j to $j+1$ is necessitated by the fact that the definition of y_j excludes one of the active bidders from consideration.

Finally, we shall state an inequality on the restart function, $r(h)$, which needs to be satisfied in order for the main theorem to hold. Consider any history h which concludes with a bidder dropping out, and let $y(h)$ be the equilibrium inference about the signal of the bidder to drop out at the conclusion of h. We require:

(14) $$r(h) \le v_{J(h)}(y(h), y(h) \mid y(h), y_{J(h)+1}, \ldots, y_{n-1})\,,$$

for all possible equilibrium histories h which conclude with a bidder dropping out. The right side of Inequality (14) is the bidding threshold *after history h* for a bidder receiving the same signal, $y(h)$, as the bidder who dropped out at the conclusion of h. Ineq. (14) thus tells us that no bidder receiving a signal $x > y(h)$ would wish to drop out of the auction at the precise moment that the clock restarts (at price $r(h)$) after history h. Since, by the definition of $y(h)$, all bidders receiving signals $x \le y(h)$ should already have dropped out of the auction by the time the clock stopped at history h, Ineq. (14) guarantees that there are no mass points in the equilibrium dropping-out behavior.

Conversely, if Ineq. (14) is not satisfied, then there exists an equilibrium history such that bidders with a nondegenerate interval of signals attempt to drop out at the moment that the clock restarts. This implies that bidders' signals cannot always be fully inferred by the price at which they drop out and, more crucially, it introduces a positive probability that two bidders attempt to drop out simultaneously in equilibrium, vastly complicating the analysis.

Finally, let us fully specify bidders' updating rules about other bidders' signals. The treatment of *unobservable* deviations is obvious: if a bidder whose true signal is x' drops out of the auction at the time which a bidder with signal x is supposed to drop out, then this bidder must be taken by her rivals to have received a signal of x. However, the treatment of *observable* deviations is not obvious, and in fact the issue does not even arise in standard auction analyses where bidders possess unit demands. As we see in eq. (12), at any moment in time, any bidder is expected to either hold constant her quantity demanded or to completely drop out of the auction. What should her rivals infer if bidder j decreases her demanded quantity somewhat but still demands more than she has already clinched? The following Updating Rule answers this question by specifying that rivals only attach informational significance to complete reductions in quantity: a partial reduction is treated the same as no reduction at all. Such an Updating Rule has some attractive intuitive appeal: in the general symmetric model under consideration, each bidder has a constant marginal value for each unit; given that bidders' demand reductions have no effect on the prices that they themselves face, the only purpose in partly reducing demand could be to deceive other bidders, so such partial reductions should be completely disregarded by rivals. The same updating rule continues to be used following both equilibrium and out-of-equilibrium histories. To be precise:

UPDATING RULE FOR $\{\beta(x,h),\gamma_i(x,h)\}_{i=1}^{n}$. As long as bidder j remains active (i.e., $q_j > C_j$), any other bidders infer that bidder j's signal is distributed according to the prior distribution (conditioned on information on the other bidders' signals), truncated from below at the lowest signal which is supposed to remain active at the current price under the bidding threshold $\beta(x,h)$. At whatever moment bidder j ceases to be active (i.e., $q_j = C_j$), other bidders update their beliefs about bidder j's signal to the lowest point remaining in the support of their (truncated) distributions and they maintain these beliefs thereafter.

We now are ready to state one of the main results of the paper:

THEOREM 3. For the general symmetric model with M objects and n bidders, who possess capacities of λ_i ($i = 1,\ldots,n$), respectively, suppose that the restart rule $r(\bullet)$ satisfies Inequality (14). Then the n-tuple of strategies $\{\beta(x,h),\gamma_i(x,h)\}_{i=1}^{n}$ defined by Eq. (12), together with the associated Updating Rule, constitutes an efficient equilibrium of the alternative ascending-bid auction.

5.3 Comparison with the Milgrom-Weber Equilibrium for a Single Object

Theorem 3, as stated above, holds for all M and all λ_i ($i = 1,\ldots,n$). Thus, as a special case, Theorem 3 must hold for the case of $M = \lambda_1 = \ldots = \lambda_n$, i.e., the single-object auction studied by Milgrom and Weber (1982a). Moreover, the alternative ascending-bid auction, when restricted to this environment, reduces to the standard English auction. Thus, it is interesting to note that the equilibrium of Theorem 3 differs somewhat from the Milgrom-Weber equilibrium: all but the penultimate bidder drop out of the auction at different bidding thresholds, but the final outcome is the same. The key to understanding the difference is to observe that Theorem 3 implicitly introduces "turning-back-the-clock" even into the single-object environment, and this somewhat alters the reasoning of bidders.

In Milgrom and Weber (1982a, Eq. (6)), when J bidders remain, bidder i with signal x stays in the auction until the price reaches:

$$(15) \qquad p = E[V_i \mid X_i = x, Y_1 = x, \ldots, Y_{J-1} = x, Y_J = y_J, \ldots, Y_{n-1} = y_{n-1}].$$

That is, bidder i bids up to her expected value for the objects, conditional on *all* of the other active bidders' signals equaling her own signal. By contrast, in Eq. (12) above, when J bidders remain, bidder i with signal x stays in the auction until the price reaches:

$$(16) \qquad p = E[V_i \mid X_i = x, Y_{J-1} = x, Y_J = y_J, \ldots, Y_{n-1} = y_{n-1}].$$

That is, bidder i bids up to her expected value for the objects, conditional on *the lowest* of the other active bidders' signals equaling her own signal.

The reason that the bidding threshold (16) does not work as an equilibrium in the Milgrom-Weber formulation is that, after other bidders have dropped out, bidder i using threshold (16) might regret staying in the auction as long as she did. With "turning-back-the-clock" specified so as to satisfy Ineq. (14) in the current article, this is no longer a problem.

If we had attempted to generalize Milgrom and Weber's bidding threshold (15) in the current article, we would have found ourselves facing a serious problem. Namely, (15) would generalize to a prescription that bidder i should bid up to her expected value for the objects, conditional on *sufficiently many* of the other active bidders' signals equaling her own signal so that she clinches objects. Such a bidding strategy behaves badly in a model where some bidders demand more than one unit. The difficulty occurs in subgames where different bidders are demanding different quantities. A bidder i demanding a larger quantity may require fewer opponents to drop out in order to clinch than a bidder j demanding a smaller quantity. Hence, it may be the case that their signals, x_i and x_j respectively, may be equal, but bidders i and j utilize unequal price thresholds in reducing their quantities. Thus, efficiency is impaired, and the overall equilibrium construction is vastly complicated. Moreover, this problem must necessarily be tackled even if the initial capacities of bidders are equal (i.e., $\lambda_1 = ... = \lambda_n > 1$), since subgames exist in which partial reductions occur, and different bidders have different capacities in such subgames.

6 The Champion's Plague: Comparison with the Vickrey Auction

Among the most famous results in the single-object auction literature is the comparison between the sealed-bid, second-price auction (the static auction) and the English auction (the associated dynamic auction). The analogous question for auctions of multiple identical objects is the comparison between the Vickrey auction and the alternative ascending-bid auction. We have already seen in Section 4 that, for the pure private values case — and similar to the single-object results — the static auction and dynamic auction are largely equivalent (although the dynamic auction may be cognitively simpler and more preserving of privacy). In this section, we extend the comparison to the general symmetric model with interdependent values. Here, we find that the dynamic auction outperforms the static auction, for two reasons. First, as in the single-object analysis of Milgrom and Weber (1982a), the dynamic auction provides greater linkage between the auction outcome and the bidders' affiliated signals, increasing the seller's expected revenues. Second, a new effect not present in the single-object analysis is discovered. The second effect will be referred to as the "Champion's Plague."

As in Ausubel and Cramton's (1996) analysis of the uniform-price auction for the general symmetric model, the analysis dichotomizes into two cases: the case where bidders have identical capacities and where the number of available objects is an integer multiple of the bidders' capacities; and the case where either bidders' capacities are unequal or where the number of available objects is not an integer multiple of the bidders' capacities. The intuitive explanation for the difference between these two cases is that when $K \equiv M/\lambda$ is an integer, the model is closely related to one in which bidders possess *unit* demands and compete for K indivisible objects. There, the Vickrey auction coincides with the $(K+1)$st price auction, and the alternative ascending-bid auction coincides with the standard ascending-bid auction. However, when $K \equiv M/\lambda$ is *not* an integer or the λ_i are *not* equal, the above auction forms do not coincide. In the first case, an equilibrium of the Vickrey auction is exhibited which falls short of the equilibrium of Theorem 3 only due to the Linkage Principle. In the second case, the Champion's Plague emerges as a factor as well.

6.1 Analysis when M/λ is an Integer: The Linkage Principle

In this section, we consider the case where $K \equiv M/\lambda$ is an integer, meaning that the number of available objects is an integer multiple of the (identical) capacities of every bidder. We will see that the results very much parallel those of Milgrom and Weber (1982a) for a single object. Despite interdependent values, the Vickrey auction, as well as the alternative ascending-bid auction, possesses efficient equilibria. However, the seller's expected revenues from the alternative ascending-bid auction (weakly) exceeds those from the Vickrey auction. The intuition is that the high bidders have more information available about the losing bidders' signals in the ascending-bid auction than in the sealed-bid auction; if bidders' signals are affiliated, this also gives the high bidders extra information about the other high bidders' signals, eroding the informational rents which any high bidder can extract.

A simple way to construct efficient equilibria of the Vickrey auction in this case is to restrict bidders to submit λ identical bids. It is straightforward to see that — in any of the auction formats considered — if each of the other bidders is utilizing a strategy consisting of submitting λ identical bids, then it is a best response for the remaining bidder to also submit λ identical bids. (However — and quite importantly — the reader should observe that the previous sentence does *not* remain true in the case where $K \equiv M/\lambda$ is *not* an integer or the λ_i are *not* equal.) In turn, equilibria satisfying the λ-identical-bid restriction exactly coincide, along the equilibrium path, with equilibria of the model where bidders with unit demands compete for K identical objects. Thus, the results when $K \equiv M/\lambda$ is an integer are almost formally equivalent to the analysis of the model where bidders with unit demands compete for K objects, and are straightforward extensions of the results for auctions of a single object.[17]

We begin by constructing an efficient equilibrium of the Vickrey auction. Let us define:

$$v_K(x,y) = E[\,V_i \mid X_i = x\,,\, Y_K = y\,]\ . \tag{17}$$

By the same reasoning as in Milgrom and Weber (1982a, Theorem 6), we have:

THEOREM 4. Let $K \equiv M/\lambda$ be an integer, and define $b^*(x) = v_K(x,x)$. Then, for the general symmetric model with n bidders each of whom has unit demand, the n-tuple of strategies $(b^*, \ldots, b^*)$ is an efficient equilibrium of the $(K{+}1)$st-price auction for K objects. Therefore, with n bidders each of whom has capacity for λ objects, the corresponding n-tuple of strategies consisting of λ identical bids is an efficient equilibrium of the Vickrey auction (as well as of the uniform-price auction) for M objects.

As we have already seen in Section 5, the alternative ascending-bid auction also exhibits an efficient equilibrium. (Indeed, in the case where $K \equiv M/\lambda$ is an integer, it seems possible that the

[17]Observe that the first parts of Theorems 4 and 5 (treating K-object auctions with unit demands) also appear in Milgrom and Weber (1982b). However, since the earlier paper remains unpublished, I include the results for completeness.

equilibrium path coincides with an equilibrium path of the standard ascending-bid auction.)[18] Analogous to the standard results for a single-object auction, the seller's expected revenues from the sealed-bid auction and the ascending-bid auction can be ranked, and the ascending-bid auction does better. By the same reasoning as in Milgrom and Weber (1982a, Theorem 11), we have:

THEOREM 5. Let $K \equiv M/\lambda$ be an integer. Then, for the K-object auction with unit demands, the efficient equilibrium of the standard ascending-bid auction raises the same or higher expected revenues as the efficient equilibrium of the (K+1)st-price sealed-bid auction. Consequently, for the M-object auction where bidders each have capacities of λ objects, the efficient equilibrium of the alternative ascending-bid auction raises the same or higher expected revenues as the efficient equilibrium of the Vickrey auction.

6.2 Analysis When M/λ is Not an Integer: The Champion's Plague

When $K \equiv M/\lambda$ is not an integer or the λ_i are not equal, we know from Ausubel and Cramton (1996, Theorems 1 and 2) that the uniform-price auction does not admit efficient equilibria, in both pure private values and interdependent values models. Surprisingly, we shall now see that when bidders' values are interdependent, the inefficiency result also extends to the Vickrey auction.

The intuition for the result is a generalization of the standard "Winner's Curse" for single-object auctions. This classic proposition of auction theory states that, in a particular informational sense, winning is "bad news." More precisely, we may state:

> THE WINNER'S CURSE. In a single-object auction with interdependent values, a bidder's expected value conditional on winning the object is less than her unconditional expected value.

Now consider instead, for example, the general symmetric model with $M = 3$ and $\lambda = 2$, i.e., an environment where each bidder equally values two objects and there are three objects available. Then, if objects are assigned efficiently, winning *one* object indicates to a bidder that her signal equaled the *second*-order statistic of all bidders' signals, while winning *two* objects indicates to a bidder that her signal equaled the *first*-order statistic of all bidders' signals. Thus, from the same informational perspective as the Winner's Curse, if winning one object is bad news, then winning two objects is worse news. For lack of better terminology, let us call this:

> THE CHAMPION'S PLAGUE. In a multiple-object auction with interdependent values, a bidder's expected value conditional on winning more objects is less than her expected value conditional on winning fewer objects.

[18] However, such a result is not the least bid obvious. Certainly, so long as bidders are constrained to demand either zero or λ units, the standard ascending-bid auction works perfectly well for $K \equiv M/\lambda$ an integer. However, it is possible that difficulties will appear in (out-of-equilibrium) subgames where $0 < q_i < \lambda$ for some bidder i.

The Champion's Plague can be easily used to show that, with interdependent values and if $K \equiv M/\lambda$ is not an integer, any equilibrium of the Vickrey auction for the general symmetric model is inefficient. The first step of the reasoning is the same as in Ausubel and Cramton (1996, Theorems 1 and 2): In order for an equilibrium of the Vickrey auction to be efficient, it must be the case that every bid of every bidder is the same function of the bidder's signal. The formal analysis is simplified if we assume that the set, B, of all allowable bids is a bounded set. Let $b_i^j(x)$ denote the jth-highest bid submitted by bidder i when her signal is x. We have the following lemma:

LEMMA 1. Suppose that an equilibrium of the Vickrey auction assigns objects to the bidders receiving the highest signals, for almost every realization. Then there exists a function $\phi(\bullet)$ from signals to bids which provides the bids of all bidders: for every bidder i ($i = 1, \ldots, n$) and bid j ($j = 1, \ldots, \lambda_i$), we have $b_i^j(x) = \phi(x)$ almost everywhere.

The second step of the reasoning is to suppose that all bidders other than i are using bid functions which satisfy Lemma 1. Then, with interdependent values, Bidder i faces the Champion's Plague, and therefore her best-response bid function is always to bid strictly less for the last unit than for the first. This establishes that no equilibrium of the Vickrey auction is efficient. We have:

THEOREM 6. Unless $\lambda_i \equiv \lambda$ ($i = 1, \ldots, n$) and M/λ is an integer, there does not exist an efficient equilibrium of the Vickrey auction.

By contrast, Theorem 3 demonstrated the existence of an efficient equilibrium in the alternative ascending-bid auction. Thus, if bidders play according to the equilibrium of Theorem 3, then the dynamic auction outperforms the static auction insofar as efficiency. The intuition for this result can again be most easily seen with $M = 3$ and $\lambda = 2$. In the efficient equilibrium of the alternative ascending-bid auction, a winning bidder clinches her first unit when the bidder with the third-highest signal drops out of the auction, and clinches her second unit not until when the bidder with the second-highest signal drops out. This division into two separate times enables a winning bidder to adjust her bidding threshold for the Champion's Plague.

Obviously, Theorem 6 should *not* be interpreted to mean that there does not exist *any* efficient static mechanism in this environment. Indeed, a straightforward way to define an efficient static mechanism is to have the bidders report their signals to a mediator, and for the mediator to carry out the precise assignments and payments which would be generated by operating the alternative ascending-bid auction. Rather, the correct interpretation of Theorem 6 is merely that the rules of the standard Vickrey auction do not properly take account of value interdependencies. In Appendix B, we will briefly develop a "generalized Vickrey auction" which does yield efficient outcomes in the face of value interdependencies. However, unlike the standard Vickrey auction or the alternative ascending-bid auction, the generalized Vickrey auction requires the auction designer to know a great deal of information about the bidders' utility functions, and requires the auction designer to change the rules whenever the bidders' specific utility functions change.

7 A Symmetric Example with Interdependent Values

One straightforward example of a model satisfying the assumptions of the general symmetric model is the following:

EXAMPLE 2. Suppose that there are n bidders, who respectively receive signals denoted X_i $(i=1,\ldots,n)$ which are independently and uniformly distributed on [0,1]. Let $\overline{X}_{-i}$ denote the arithmetic average of X_{-i}. We can then consider: $V_i = \beta X_i + (1-\beta)\overline{X}_{-i}$, for each i $(i=1,\ldots,n)$, where $\beta \in (1/n,1)$.

Observe that, as $\beta \to 1$, this example approaches a model with independent private values and, as $\beta \to 1/n$, this example approaches a model with pure common values. However, here, we consider the continuum of cases in between. Also observe that the (independent) signals in Example 2 satisfy only weak affiliation and not strict affiliation, so therefore the weak inequality in Theorem 5 may only be satisfied with equality.

Let us now develop equilibrium bidding rules for the alternative ascending-bid auction, for the special case where $n = 3$, $M = 3$, and $\lambda = 2$. That is, a seller offers three units of a homogeneous good to three bidders, each of whom has a constant marginal value for up to two units. If the auctioneer begins with a price of zero, all three bidders will indicate that they are "in" for two units each. As the auctioneer raises the price, the bidder who received the lowest signal will drop out at a price which we may denote p_3. At this point, each of the remaining two bidders "clinch" winning one unit each at a price of p_3. The auctioneer then "turns back the clock," for example, to zero, and resumes naming prices. As the auctioneer raises the price, the bidder who received the middle-valued signal will drop out at a price which we may denote p_2. At this point, the remaining bidder (who received the highest signal) clinches a second unit at a price of p_2, and the auction concludes. Observe that this symmetric equilibrium yields an efficient outcome: If $x_1 > x_2 > x_3$, then $v_1 > v_2 > v_3$. We calculate:

$$E[V_i \mid X_i = x, Y_2 = x] = \beta x + ½(1-\beta)[(1+x)/2 + x] = [(3+\beta)/4]x + (1-\beta)/4.$$

$$E[V_i \mid X_i = x, Y_1 = x, Y_2 = y_2] = \beta x + ½(1-\beta)(x + y_2) = ½(1+\beta)x + ½(1-\beta)y_2.$$

These calculations thus yield:

PROPOSITION 1. For Example 2 with $n = 3$, $M = 3$, and $\lambda = 2$, the alternative ascending-bid auction with full bid information possesses an efficient equilibrium consisting of the following strategies:

- Before any bidder has dropped out, Bidder i $(i=1,2,3)$ stays in for two units until such time that the price equals $[(3+\beta)/4]x_i + (1-\beta)/4$; if the price reaches this level before any other bidder has dropped down to zero units, Bidder i then drops to zero units.
- After one other bidder has dropped out at a price p_3, Bidder i $(i=1,2,3)$ calculates that bidder's implied signal, x_d, by $[(3+\beta)/4]x_d + (1-\beta)/4 = p_3$. Bidder i stays in until such time that the price equals $½(1+\beta)x_i + ½(1-\beta)x_d$; if the price reaches this level before the other remaining bidder has dropped out, Bidder i then drops out.

By contrast, Theorem 2 of Ausubel and Cramton (1996) and Theorem 6 above imply the following second proposition:

PROPOSITION 2. For Example 2 with $n = 3$, $M = 3$, and $\lambda = 2$, there do not exist efficient equilibria of either the uniform-price auction or the Vickrey auction.

8 Conclusion

This article has proposed a new ascending-bid auction format for multiple objects, and has shown that it yields efficient outcomes when a multi-unit generalization of the Milgrom and Weber (1982a) assumptions is satisfied. In the formal analysis, the model was assumed to be symmetric among bidders, in the sense that their signals were drawn from the same distribution and their values were given by the same function of their own and others' signals, and the model was assumed to be symmetric among objects, in the sense that a bidder's marginal value for each additional unit (up to the bidder's capacity) was constant. The first observation with which I conclude this article is to note that these assumptions, while *sufficient* for efficiency, are hardly *necessary*. Indeed, an interesting research question is to characterize exactly what minimal set of assumptions leads to efficient outcomes in the new auction.

As a straightforward example of an asymmetric model where efficiency still holds, consider the following scenario, which extends a single-object example of Maskin (1992, p. 127, footnote 1).

EXAMPLE 3. Suppose that there are three bidders, who respectively receive signals denoted X_i $(i = 1, 2, 3)$ which are independently and uniformly distributed on [0,1]. Their signals map to constant marginal valuations according to the functions:

$$V_1 = X_1 ,$$

$$V_2 = X_2 ,$$

$$V_3 = X_3 + 2/3\, X_1 + 2/3\, X_2 .$$

The number of objects, M, and the bidders' respective capacities, λ_i $(i = 1, 2, 3)$, are arbitrary.

Observe that, in Example 3, it can never be the case that Bidder 3 has the lowest valuation, as $V_3 > ½(V_1 + V_2)$. Consequently, it is straightforward to see that the following strategies form an efficient equilibrium:[19]

[19]The indicated bidding thresholds are calculated according to the same principle as in Section 5: Any bidder bids up to her expected value for the object, conditional on the lowest of the other active bidders' values equaling her own value. (In addition, Bidder 3 never updates her beliefs on X_1 or X_2 above 1, the top of the distribution's support.) The calculation is trivial for Bidders 1 and 2, as their values are purely private. Observe that, in equilibrium, either Bidder 1 or Bidder 2 drops out before the price reaches 1, so Bidder 3 is never the first to drop out. (However, since she never updates above 1, she never estimates her value at greater than $4/3 + X_3$.) Without loss of generality, suppose that Bidder 1 is the first to drop out, at price p_1; the remaining bidders infer that $X_1 = p_1$. Bidder 3 thus calculates her value to be $2/3p_1 + 2/3X_2 + X_3$. Bidder 3's value equals Bidder 2's value if: $X_2 = 2/3p_1 + 2/3X_2 + X_3$, which implies $X_2 = 2p_1 + 3X_3$. Thus, conditional on Bidder 2's value equaling her own, Bidder 3's value equals $2/3p_1 + 2/3(2p_1 + 3X_3) + X_3 = 2p_1 + 3X_3$, her bidding threshold in equilibrium play. (However, since she never updates above 1, she never estimates her value at greater than $2/3p_1 + 2/3 + X_3$.)

(18) BEFORE ANY BIDDER HAS DROPPED OUT:

Bidder 1 maintains a quantity $q_1 = \lambda_1$ until $p = X_1$, and then drops out.

Bidder 2 maintains a quantity $q_2 = \lambda_2$ until $p = X_2$, and then drops out.

Bidder 3 maintains a quantity $q_3 = \lambda_3$ until $p = 4/3 + X_3$, and then drops out.

AFTER BIDDER i ($i = 1$ OR 2) HAS DROPPED OUT AT PRICE $p_i \leq 1$:

Bidder j ($j = 3 - i$) maintains a quantity $q_j = \lambda_j$ until $p = X_j$, and then drops out.

Bidder 3 maintains a quantity $q_3 = \lambda_3$ until $p = \min\{2p_i + 3X_3\, , 2/3\, p_i + 2/3 + X_3\}$, and then drops out.

It is easy to verify that, given bidding strategies (18), the bidders drop out in ascending order of value, for all possible realizations of (X_1, X_2, X_3). Moreover, observe that the "turning back the clock" of Section 3.3 is never required here. Thus, given any number of objects and any bidder capacities, the new ascending-bid auction format yields an efficient allocation.

By way of contrast, none of the standard sealed-bid auction formats could yield efficiency for Example 3. Consider a realization of $X_2 = 5/6$ and $X_3 = 1/6$. Whether Bidder 2 or Bidder 3 should first be assigned objects depends, respectively, on whether X_1 is less than 1/6 or X_1 is greater than 1/6. However, in any sealed-bid format, Bidders 2 and 3 cannot distinguish between a state of the world where $X_1 = 1/6 - \varepsilon$ and a state of the world where $X_1 = 1/6 + \varepsilon$, and they submit the same bids in either of those states. Moreover, under standard auction rules, the bid of Bidder 1 is an irrelevant low bid which does not influence the assignment of objects to Bidder 2 versus Bidder 3.

Nor can the standard ascending-bid auction yield efficiency, unless $\lambda_1 = \lambda_2 = \lambda_3 \equiv \lambda$ and M/λ is an integer (Ausubel and Cramton, 1996, Theorem 2).

A second observation to make is that I have considered models herein with the property that bidders either have pure private values or effectively a combination of private values and common values. It is also interesting to compare the equilibrium outcomes of various auction formats when bidders have pure common values. With pure common values, all assignments of the objects are allocatively efficient; however, different auction rules may yield different seller revenues. We perform a comparison of various auction formats in Ausubel and Cramton (1997).

A final observation with which I conclude this article is that the proposed auction format is conducted with a clock: the auctioneer successively announces prices, and the bidders respond with quantities. It should be possible to obtain similar results if, instead of using the clock, one proceeds in the mode of the FCC auctions and allows the participants to name their own bids.[20] This is the subject of continuing research.

[20]Obviously, substantive changes in the auction rules are required from the rules of the actual FCC auctions, given that (as we have seen in Section 2) the FCC rules lent themselves through arbitrage to outcomes with a uniform-price character.

APPENDIX A

PROOF OF THEOREM 1.

At any point in the alternative ascending-bid auction up until its end, all of the payoff-relevant events in the auction occur through clinching. The cumulative quantity of clinched units for bidder i at price p^ℓ is given by Eq. (2). Observe that the right side of Eq. (2) is independent of bidder i's actions, except through the implicit requirement that $q_i^\ell > M - \sum_j q_j^\ell$ for $\ell = 0, \ldots, L-1$; hence, changing one's own bid strategy can have no effect on payoff, except to the extent that (1) it leads rival bidders to respond; or (2) it determines one's own final quantity q_i^*.

With no bid information, a rival bidder cannot distinguish between two strategies of bidder i, except to the extent that one or the other ends the auction. Hence, changing one's own bid strategy cannot lead rival bidders to respond. Therefore, bidder i maximizes her payoff in all states of the world by simply taking the clinching process of Eq. (2) as exogenous and selecting a strategy which optimally selects q_i^*. Observe that bidder i strictly prefers clinching a kth object at clock price p over not clinching if and only if $V_{i,k} > p$. Since marginal utilities were assumed (weakly) diminishing, we conclude that sincere bidding is a weakly-dominant strategy in this auction and that all bidders using sincere bidding is an efficient equilibrium. ■

PROOF OF THEOREM 2.

For either full bid information or aggregate bid information, let Σ_0 denote the class of strategy n-tuples such that bidders' quantities depend only on the current price, and *not* on previous quantity reductions of other bidders. Observe, by the same reasoning as in the proof of Theorem 1, that sincere bidding is a best response to the limited class of strategies Σ_0. Since sincere bidding itself is an element of Σ_0, this establishes that sincere bidding by all bidders is an equilibrium. ■

PROOF OF THEOREM 3.

By its definition in Eqs. (11) and (12), the bidding threshold $\beta(x,h)$ is increasing in the signal x, for each history h. Hence, if bidders $N \backslash \{i\}$ play according to $\beta(x,h)$ and if bidder i clinches a positive number of units due to one of J active bidders decreasing her quantity, then the quantity-decreasing bidder was using a bidding threshold of $v_J(Y, Y \mid y_J, \ldots, y_{n-1})$, where Y denotes the quantity-decreasing bidder's own signal. Evaluated at $Y = y_{J-1}$, this yields a price for bidder i of:

$$(18) \qquad v_J(y_{J-1}, y_{J-1} \mid y_J, \ldots, y_{n-1}) = E[V_i \mid X_i = y_{J-1}, Y_{J-1} = y_{J-1}, Y_J = y_J, \ldots, Y_{n-1} = y_{n-1}].$$

Meanwhile, bidder i's conditional expectation of V_i given $X_i, Y_{J-1}, Y_J, \ldots, Y_{n-1}$ is given by:

$$(19) \qquad v_J(x, y_{J-1} \mid y_J, \ldots, y_{n-1}) = E[V_i \mid X_i = x, Y_{J-1} = y_{J-1}, Y_J = y_J, \ldots, Y_{n-1} = y_{n-1}].$$

Combining Eqs. (18) and (19), and using the fact that $v_J(x, y_{J-1} \mid y_J, \ldots, y_{n-1})$ is increasing in x, we see that bidder i's conditional expected payoff on the clinched units is positive whenever $x > y_{J-1}$ and is negative whenever $x < y_{J-1}$. [This can also be seen by expanding the expectations operator, as in the proof of Theorem 4, below.] Observe that, if bidder i herself uses strategy $\beta(x,h)$, bidder i will clinch objects whenever $X_i > Y_{J-1}$ and will not clinch objects whenever $X_i < Y_{J-1}$, allowing us to conclude that $\beta(x,h)$ is a best response for bidder i providing that clinching is due to *one* of J active bidders decreasing her quantity. Moreover, bidder i can disregard the possibility that clinching is due to *two or more* bidders decreasing their quantities: using the facts that each bidding threshold $\beta_i(x,h)$ is strictly increasing in signal x, that Ineq. (14) is satisfied, and that the joint distribution of signals has no mass points, we see that simultaneous reduction is a zero-probability event. In addition, a partial decrease in quantity demanded cannot be a profitable deviation for bidder i, as it does not prompt bidders $N \setminus \{i\}$ to alter their bidding thresholds. The above argument holds for every possible J, establishing that $\beta(x,h)$ is a best response for bidder i after all histories. ■

PROOF OF THEOREM 4.

We will assume that bidders $N \setminus \{i\}$ play according to the bid function $b^*(x)$, and we will then demonstrate that $b^*(x)$ is the optimal response for bidder i. By assumption A.1, $b^*(\bullet)$ is a strictly increasing function. Hence, bidder i wins an object if and only if she bids $b > b^*(Y_K)$ (ignoring ties, which are zero-probability events). Moreover, whenever bidder i wins an object, she pays a price of $b^*(Y_K)$. Similar to the reasoning in Milgrom and Weber (1982a, Theorem 6), bidder i's expected payoff from bidding b when her signal is x is therefore:

$$E\{[V_i - b^*(Y_K)]\,1_{\{b^*(Y_K) < b\}} \mid X_i = x\} = E\{E\{[V_i - b^*(Y_K)]\,1_{\{b^*(Y_K) < b\}} \mid X_i, Y_K\} \mid X_i = x\} =$$

$$= E\{[v_K(X_i, Y_K) - v_K(Y_K, Y_K)]\,1_{\{b^*(Y_K) < b\}} \mid X_i = x\} = \int_{-\infty}^{b^{*-1}(b)} [v_K(x,z) - v_K(z,z)]\, dF_{Y_K}(z|x)\,,$$

where $dF_{Y_K}(z|x)$ denotes integration against the conditional distribution of Y_K given $X_i = x$. Again by assumption A.1, the integrand is positive for $z < x$ and negative for $z > x$. Hence, the integral is (uniquely) maximized by choosing b so that $b^{*-1}(b) = x$, i.e., $b = b^*(x)$. This establishes that $b^*(\bullet)$ is the (unique) best response for bidder i, as required. ■

PROOF OF THEOREM 5.

As a minor variation on the notation of eq. (11), let us define: $w(x, y; y_{K+1}, \ldots, y_{n-1}) = E[V_i \mid X_i = x, Y_K = y, Y_{K+1} = y_{K+1}, \ldots, Y_{n-1} = y_{n-1}]$. If all n bidders use the strategies defined by Eq. (12) (and by the symmetry assumed in A.1 and A.4), the seller's expected revenue in the standard or alternative ascending-bid auction is given by $E\{w(Y_K, Y_K; Y_{K+1}, \ldots, Y_{n-1}) \mid X_i > Y_K\}$. Meanwhile, if all n bidders use the strategy defined in Theorem 4, the seller's expected revenue in the $(K+1)$st-price or uniform-price auction is given by $E\{v_K(Y_K,Y_K) \mid X_i > Y_K\}$. Closely following Milgrom and Weber (1982a, Theorem 8), we will now demonstrate that the first quantity is at least as great as the second quantity. Observe that, if $x > y$, then:

$$\begin{aligned} v_K(y,y) &= E[V_i \mid X_i = y,\ Y_K = y] \\ &= E\{E[V_i \mid X_i = x, Y_K = y, Y_{K+1} = y_{K+1}, \dots, Y_{n-1} = y_{n-1}] \mid X_i = y,\ Y_K = y\} \\ &= E[w(X_i, Y_K; Y_{K+1}, \dots, Y_{n-1}) \mid X_i = y,\ Y_K = y] \\ &= E[w(Y_K, Y_K; Y_{K+1}, \dots, Y_{n-1}) \mid X_i = y,\ Y_K = y] \\ &\le E[w(Y_K, Y_K; Y_{K+1}, \dots, Y_{n-1}) \mid X_i = x,\ Y_K = y]\ . \end{aligned}$$

Consequently, taking the conditional expectation of each side of this inequality, given $X_i > Y_K$, yields:

$$\begin{aligned} E\{v_K(Y_K,Y_K) \mid X_i > Y_K\} &\le E\{E[w(Y_K,Y_K;Y_{K+1},\dots,Y_{n-1}) \mid X_i, Y_K] \mid X_i > Y_K\} \\ &= E\{w(Y_K,Y_K;Y_{K+1},\dots,Y_{n-1}) \mid X_i > Y_K\}\ . \end{aligned}$$

This inequality establishes that the seller's expected revenue from the dynamic auctions is at least the seller's expected revenue from the static auctions, as required. ■

PROOF OF LEMMA 1.

Let m denote the greatest integer such that $m\lambda < M$. First, we will demonstrate that any bidder must use the same bid for all quantities, almost everywhere in signals. If objects are assigned to the bidders receiving the highest signals, $q_i^* = \lambda$ if $x_i > x_{(m+1)}$ and $q_i^* = 0$ if $x_i < x_{(m+1)}$, where $x_{(m+1)}$ denotes the (m+1)st-order-statistic of *all* bidders' signals. Hence, for any $x > x'$, $b_i^\lambda(x) \ge b_i^1(x')$. Otherwise, when $x' < x_{(m+1)} < x$, x must win λ and x' must win 0, but this cannot happen if $b_i^\lambda(x) < b_i^1(x')$. Defining $B_i(x) = [b_i^\lambda(x),\ b_i^1(x)]$, this implies that $B_i(\cdot)$ is a weakly-increasing correspondence. Also define $\Delta_i(x) = b_i^1(x) - b_i^\lambda(x)$, and $Z_i = \{x \mid \Delta_i(x) > 0\}$. Thus, Z_i is the set of all x such that bidder i's bids for all quantities are not the same. Since higher bids are accepted before lower bids, $\Delta_i(x) \ge 0$ for all x. Since all bids $b_i^j(\bullet) \in B$, a bounded set, there can be at most countably many x such that $\Delta_i(x) > 0$, and so the measure of Z_i equals zero, for all $i = 1,\dots,n$. We may thus define $\phi_i(x)$ so that $b_i^j(x) = \phi_i(x)$ for all $j = 1,\dots,\lambda$ and all $x \notin Z_i$.

Second, we will demonstrate that all bidders use the same bid function, almost everywhere in signals. Otherwise, and using the fact that a monotonic function is continuous almost everywhere, there exist bidders i and h ($i \ne h$) and signals $x_i \notin Z_i$ and $x_h \notin Z_h$ such that $x_i > x_h$ but $\phi_i(x_i) < \phi_h(x_h)$. But then, consider any realization of $x_1,\dots,x_n$ such that $x_i = x_{(m)}$ and $x_h < x_{(m+1)}$ (this occurs with strictly positive probability). Assignment of the objects to the bidders receiving the highest signals requires that i win λ and h win 0. But this cannot happen, since $\phi_i(x_i) < \phi_h(x_h)$. Hence, we conclude that the bids are constant in signals: $b_i^j(x) = \phi(x)$ for almost every x.

Similar reasoning applies if the λ_i are unequal. ■

PROOF OF THEOREM 6.

Suppose that an efficient equilibrium of the Vickrey auction exists. By Lemma 1, each bidder must use the same bid function for quantities $1, \ldots, \lambda$. Suppose that M/λ is not an integer. Let m denote the greatest integer such that $m\lambda < M$. By the usual reasoning in the auctions literature, bidder i's bid for her last unit is $b_i^\lambda(x) = E[V_i \mid X_i = x, Y_m = y]$, since bidder i wins λ units if and only if her signal is at least the mth order statistic of the other bidders. However, bidder i's bid for her first unit is $b_i^1(x) = E[V_i \mid X_i = x, Y_{m+1} = y]$, since bidder i wins 1 unit if and only if her signal is at least the $(m+1)$st order statistic of the other bidders. Thus, $b_i^\lambda(x) < b_i^1(x')$, counter to Lemma 1, and yielding a contradiction to the existence of an efficient equilibrium.

Similar reasoning can be applied to the bidder(s) i with the largest capacity λ_i, if the λ_i are unequal. ■

APPENDIX B

A GENERALIZED VICKREY AUCTION.

Given the focus of Sections 5 and 6 on the dynamic auction with interdependent values (and where signals are affiliated), it seems useful to specify a static mechanism which can serve as a baseline for comparison. As established by Theorem 6, the Vickrey auction does not provide an appropriate baseline, since the Vickrey auction does not yield an efficient allocation. However, for the case of a single object where bidders have interdependent values, Maskin (1992) defined a "modified second-price auction" which is a useful baseline of comparison for the English auction. In the same spirit, in this Appendix, we shall generalize Maskin's approach by defining a "generalized Vickrey auction" for multiple objects with interdependent values.[21] The generalized Vickrey auction will provide an appropriate static mechanism for comparison with the alternative ascending-bid auction.

Consider the following generalization of the model of Section 5.1. Let $x \equiv (x_1, \dots, x_n)$ denote the vector of n real-valued signals which are privately observed by the n respective bidders. Also let x_{-i} denote the $(n-1)$ signals observed by all bidders except i. Now let $u_i^k(x_i, x_{-i})$ denote the marginal value of a kth unit for bidder i, given bidder i's own signal and her opponents' signals. We assume that $u_i^k(\bullet)$ is continuous and nondecreasing in its arguments. We will not require symmetry of bidders for the treatment in this Appendix, but an interesting special case of the allowed specification is if $u_i^k(\bullet)$ is the same nonnegative-valued function for each i $(i = 1, \dots, n)$, i.e., symmetric bidders.

Now let us extend the single-object treatment of Maskin (1992, p. 27) by defining the following direct mechanism. Each bidder i $(i = 1, \dots, n)$ reports her type x_i to a mediator. Given the n reports, the mediator determines the allocation $(K_1, \dots, K_n)$ of quantities to the n bidders which maximizes surplus. Associated with the kth object assigned to bidder i $(1 \le k \le K_i)$, we define $x_i^k(x_{-i})$ to be the lowest signal so that, if signal x_i^k is reported by bidder i and if the vector of signals x_{-i} is reported by bidder i's opponents, then bidder i receives at least k units in the efficient allocation. Finally, the payment rule is defined as the following modification to the payment rule of the standard Vickrey auction: bidder i pays the kth highest rejected value (other than her own) for her kth object, where values are evaluated for this calculation using $x_i^k(x_{-i})$ as the signal for bidder i and using x_{-i} (the vector of actual reports) as the signals for bidder i's opponents.

Observe that this static mechanism has the same general flavor as the Vickrey auction. Any bidder's submitted bid does not determine the price she pays (conditional on winning the object), since: (1) à la Vickrey, her payment is determined only by the opportunity cost of providing her with the object; and (2) in computing the opportunity cost, the bidder's actual reported signal is not used, but rather the lowest signal which would enable her to win the object.

[21]Contemporaneous research by Dasgupta and Maskin (1997) yields an auction mechanism which, for the case of multiple identical objects, appears to be outcome-equivalent to the modified Vickrey auction of this Appendix.

For the symmetric model treated in Sections 5 and 6, it is easy to verify that the generalized Vickrey auction induces sincere bidding, and hence yields an efficient allocation. Thus, it is an appropriate benchmark for comparison with the alternative ascending-bid auction.

Indeed, in any environment where bidders' signals are strictly affiliated and where the efficient allocation assigns a positive number of units to two or more bidders, the generalized Vickrey auction outperforms the alternative ascending-bid auction for the symmetric model, in the sense of yielding an equally efficient outcome but generating higher seller revenues. The explanation for this is quite simple. At the time that units are first "clinched" in the alternative ascending-bid auction, the private signals of two or more bidders have not yet been revealed, and so the payment is based on at most $(n-2)$ private signals. By contrast, in the generalized Vickrey auction, all of the private signals have been revealed to the mediator, and the payment is then allowed to depend on $(n-1)$ private signals. By the same argument as in Milgrom and Weber (1982), the latter auction uses more private signals and hence yields higher expected revenues.

However, the generalized Vickrey auction also has a serious disadvantage. Paraphrasing Maskin (1992, p. 127, footnote 3): Notice, however, that the rules of the auction are *defined* in terms of the functions $u_i^k(\bullet)$. That is, the auction designer must know these functional forms, a demanding requirement. By contrast, the designer can be ignorant of the forms if he uses the alternative ascending-bid auction.

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PRICE EXCEEDED	QUANTITIES DEMANDED						AGGREGATE DEMAND	CLINCHING OF LICENSES
	Bidder A	*Bidder B*	*Bidder C*	*Bidder D*	*Bidder E*	*Bidder F*		
10	3	1	3	2	2	1	12	
25	3	1	3	2	1	1	11	
45	3	1	3	2	0	1	10	
49	3	1	2	2	0	0	8	
65	3	1	2	1	0	0	7	A “clinches“ a license
75	3	0	2	1	0	0	6	A & C “clinch” licenses
85	3	0	2	0	0	0	5	A & C “clinch” licenses

TABLE 1

Progression of Bidding in Alternative Ascending-Bid Auction
for Example 1

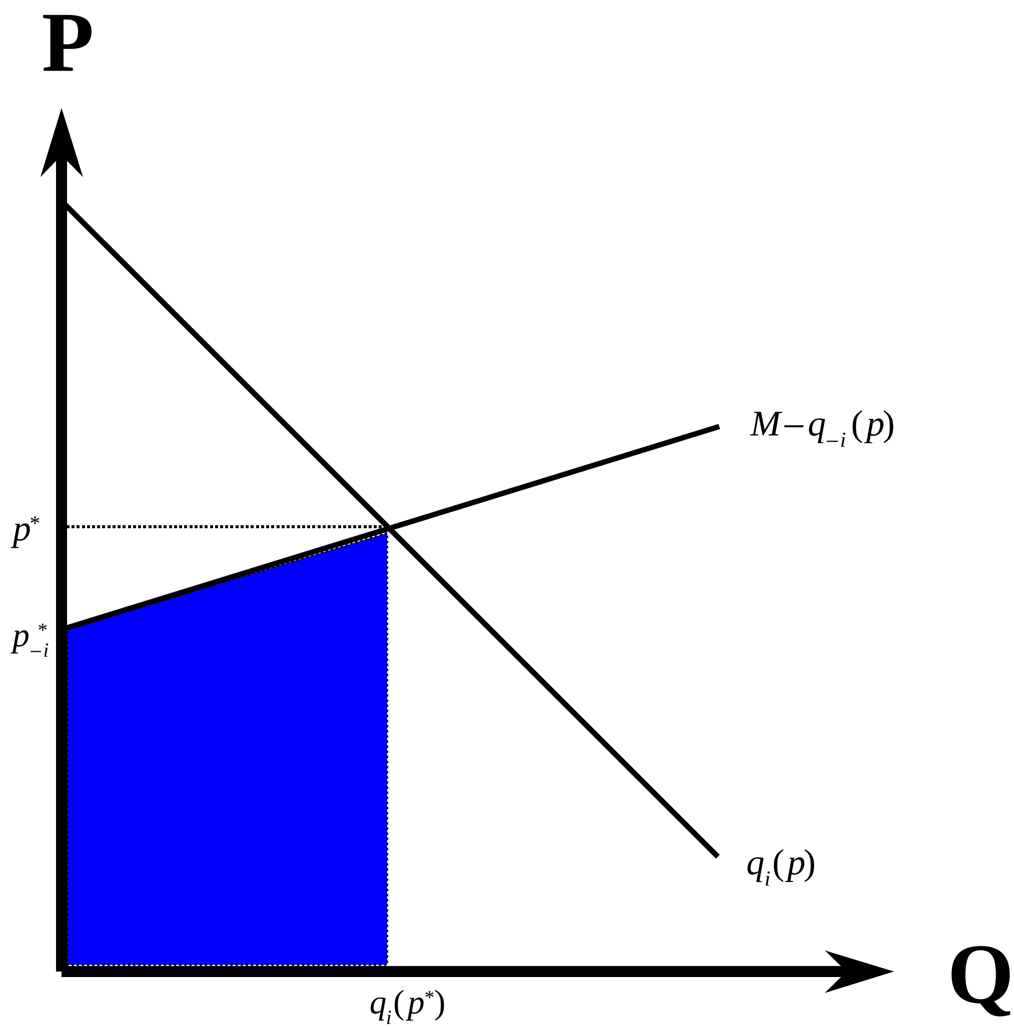

Figure 1

Payment Rule in the Vickrey Auction