# AN EFFICIENT ASCENDING-BID AUCTION FOR DISSIMILAR OBJECTS

by

Lawrence M. Ausubel*

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*Department of Economics, University of Maryland, College Park, MD 20742; (301) 405-3495; ausubel@econ.umd.edu.

## 1. Introduction

One of the most compelling advantages of the English auction over the sealed-bid, second-price auction is that it protects the bidder possessing the highest value from needing to ever reveal her value to the seller and to other bidders (Rothkopf, Teisberg, and Kahn, 1990; Engelbrecht-Wiggans and Kahn, 1991; Rothkopf and Harstad, 1995). Suppose that a broadcast license were to be sold by second-price, sealed-bid auction. Say that Bidder A, who valued the license the most, placed a value of $200 million on the license, while Bidder B, the second-highest-valuation buyer, placed a value of only $50 million on the license. Assuming independent private values, observe that the dominant-strategy equilibrium in the sealed-bid, second-price auction requires each bidder to submit a sealed bid equaling her true value. However, bidders may fear the following scenario. The seller, knowing after the bidding that Bidder A actually values the license at $200 million, may attempt to renege on the sale, and renegotiate the price above the $50 million established by the auction. Alternatively, the seller, after receiving the $200 million sealed bid, may surreptitiously plant a bogus $199 million bid (or enlist a "shill" to insert a bid in his own name). If the seller is the Government, the seller may fear the public-relations disaster when it becomes generally known that it is selling a public asset which Bidder A values at $200 million for a price which is a mere quarter of that value.[1] Finally, there are business reasons why Bidder A may wish to conceal the fact that her value is so high, for example if she is contemplating buying additional broadcast licenses, either from the Government, through subsequent auctions, or from private parties, through negotiations.

By contrast, an English auction avoids this problem. With the valuations described above, Bidder A is only required to reveal in the auction process that she values the license at greater than $50 million. The fact that her true threshold equals $200 million never needs to be elicited. Hence, the seller cannot make opportunistic use of Bidder A's true value to drive up the price, the seller is spared the public embarrassment of failing to capture the difference between the first- and second-highest values, and the highest buyer maintains the secrecy of her value for use in future transactions. Regretably, the exact value of Bidder B — unlike that of Bidder A — is revealed to the seller in the course of the auction, but ascertaining the second-highest-bidder's valuation

[1] McMillan (1994, p. 148) notes that this became a *bona fide* problem for the New Zealand government when it adopted second-price, sealed-bid auctions for spectrum licenses in the early 1990's.

seems to be an inevitable part of placing the license in the hands who value it the most.

The purpose of the current paper is to propose a new ascending-bid auction scheme for selling multiple, dissimilar objects, which has the analogous advantage of conserving on the revelation of high-bidders' values. It begins with the Vickrey auction for multiple, dissimilar objects (often also known as the Groves mechanism, Groves-Clark mechanism, or Groves-Ledyard mechanism), but transforms it into a progressive procedure which stops eliciting information the moment that no further information is needed to determine the efficient allocation. In the language of the analogy questions contained in standardized college admission tests: second-price auction is to English auction, as Vickrey auction is to the auction proposed in this paper.

The existing paper most closely related to the current paper is my earlier manuscript (Ausubel, 1995) which treats auctions for multiple, identical objects. The earlier manuscript's alternative auction — which may be viewed as a special case of the current auction design — exploits features of the homogeneous-good environment to construct an eminently-simple dynamic procedure. Unfortunately, the case of dissimilar objects does not lend itself to so simple a procedure. The reason for the difference in complexity is immediately identifiable from a cursory look at the corresponding Vickrey auctions for the two environments. With multiple identical objects, all the information which the auctioneer must extract is each bidder's value associated with every possible quantity of the good. If bidders exhibit diminishing marginal values, then it is straightforward for the auctioneer to obtain this information using a single ascending clock in marginal values, and it is unnecessary to run this clock above the marginal value at which the market clears. By contrast, with multiple dissimilar objects, the Vickrey auctioneer needs to extract each bidder's value for every possible subset of the set of objects being auctioned. In some sense, this necessitates utilizing a multiplicity of ascending clocks, each to obtain information concerning differences in bidders' value between one subset of objects and another. The current paper presents one specific procedure for extracting this information in a sequential fashion. The procedure is guaranteed to terminate in finite time, and at an efficient allocation of the objects being auctioned. The procedure yields truthful revelation as a weakly-dominant strategy, and will generally conserve on the revelation of values of the highest-valuation bidder. It is not intended to be the unique specific procedure which possesses these properties; rather it is intended to simply demonstrate the existence of a procedure with these properties, and to demonstrate the desirability of yielding (generating, obtaining) these properties.

## 2. Two Illustrative Examples, Involving Two Bidders and Two Dissimilar Objects

EXAMPLE 1:

Suppose that two dissimilar — but somewhat related — broadcast licenses, denoted A and B, are offered simultaneously for auction.[2] Each bidder possesses a value for each license separately, and for the two licenses together. It is assumed that each bidder's values for these licenses are additively separable from their values for everything else in the world, that these values are expressible in monetary units, and that we normalize to zero the value associated with possessing neither license. There are two bidders with values in the relevant range, and their values are given as follows (where numbers are expressed in millions of dollars):

(2.1) *Bidder 1*:

$$v_1(\emptyset) = 0$$
$$v_1(\{A\}) = 200$$
$$v_1(\{B\}) = 60$$
$$v_1(\{A,B\}) = 260 \;;$$

*Bidder 2*:

$$v_2(\emptyset) = 0$$
$$v_2(\{A\}) = 40$$
$$v_2(\{B\}) = 40$$
$$v_2(\{A,B\}) = 50 \;.$$

In this example, bidders are presumed to possess complete information about their rivals' valuations, but exactly the same logic would apply if they possessed independent private values.

In the Vickrey auction for this situation, each bidder would submit a sealed bid consisting of a price associated with each subset of the available objects, i.e., for each of $\emptyset$, {A}, {B} and {A,B}. The auctioneer would then determine which allocation of goods is associated with the highest total bids; in this example, assigning both licenses A and B to Bidder 1 yields the highest bids, totaling 260. However, Bidder 1 does not pay her bid of 260. Instead, the auctioneer also calculates the allocation of goods associated with the highest total bids *if Bidder 1 were absent*

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[2] The reader may observe that Examples 1 and 2 bear some superficial resemblance to the FCC's upcoming auction for Direct Broadcast Satellite (DBS) licenses, currently scheduled for January 24, 1996. While the DBS auction does, in fact, involve the sale of two dissimilar licenses, it is probably not a prime candidate for the auction design proposed in this paper. In any case, the bidder valuations chosen for Examples 1 and 2 were chosen without reference to or knowledge of bidders' true values in the DBS auction.

*from the bidding*, thus determining the marginal surplus which Bidder 1 brings to the auction. The auctioneer then requires a payment by Bidder 1 chosen so that the surplus obtained by Bidder 1 exactly equals the marginal surplus which Bidder 1 brings to the auction. In this example, the total bids associated with the optimal allocation of the objects, in the absence of Bidder 1, equals 50. Thus, Bidder 1's payment for the two licenses is set equal to 50, and then the surplus of 210 she obtains exactly equals the marginal surplus of 210 which she brings to the auction. As is now well known, sincere bidding is a weakly-dominant strategy in the Vickrey auction (Vickrey, 1961; Clark, 1971; Groves, 1973; Groves and Ledyard, 197_). With values that are complete information, or with independent private values, it is weakly dominant for each bidder to submit a sealed bid exactly equaling his true values from Eqs. (2.1), and the Vickrey auction is then guaranteed to assign an efficient allocation of the good.

It should now briefly be observed that it is unnecessary for the auctioneer to learn that Bidder 1 values {A,B} at 260, in order for the auctioneer to implement the outcome of the Vickrey auction. It will suffice for the auctioneer to determine the following facts: (i) the licenses are most efficiently awarded to a single bidder; (ii) $v_2(\{A,B\}) = 50$; and (iii) $v_1(\{A,B\}) > 50$. Once these three facts are elicited, the auctioneer can call an immediate end to the auction and carry out the Vickrey outcome. This can be done while maintaining the confidentiality of Bidder 1's value for the two licenses together (and avoiding the potential ills discussed in the Introduction), thus making it more likely that Bidder 1 will feel confident enough to be willing to reveal her true value of 260.

Let us now see how my design for an ascending-bid auction of dissimilar objects would proceed, and why it would accomplish these goals. The easiest conceptualization of the procedure is to think of six separate auctions being run simultaneously, each of which in turn consists of two or three subauctions also being run simultaneously, as follows:

AUCTION I: "BIDDER 1'S AUCTION FOR {A,B}"

An ascending-bid auction is conducted between Bidder 1 and Bidder 2, for each of the following differences in value:

| | | | |
|---|---|---|---|
| A. | $v_1(\{A,B\}) - v_1(\varnothing)$ | *versus* | $v_2(\{A,B\}) - v_2(\varnothing)$ |
| B. | $v_1(\{A,B\}) - v_1(\{A\})$ | *versus* | $v_2(\{B\}) - v_2(\varnothing)$ |
| C. | $v_1(\{A,B\}) - v_1(\{B\})$ | *versus* | $v_2(\{A\}) - v_2(\varnothing)$ |

AUCTION II: "BIDDER 1'S AUCTION FOR {A}"

An ascending-bid auction is conducted between Bidder 1 and Bidder 2, for each of the following differences in value:

| | | | |
|---|---|---|---|
| A. | $v_1(\{A\}) - v_1(\varnothing)$ | *versus* | $v_2(\{A,B\}) - v_2(\{B\})$ |
| B. | $v_1(\{A\}) - v_1(\{B\})$ | *versus* | $v_2(\{A\}) - v_2(\{B\})$ |

AUCTION III: "BIDDER 1'S AUCTION FOR {B}"

An ascending-bid auction is conducted between Bidder 1 and Bidder 2, for each of the following differences in value:

| | | | |
|---|---|---|---|
| A. | $v_1(\{B\}) - v_1(\varnothing)$ | *versus* | $v_2(\{A,B\}) - v_2(\{A\})$ |
| B. | $v_1(\{B\}) - v_1(\{A\})$ | *versus* | $v_2(\{B\}) - v_2(\{A\})$ |

AUCTION IV: "BIDDER 2'S AUCTION FOR {A,B}"

An ascending-bid auction is conducted between Bidder 2 and Bidder 1, for each of the following differences in value:

| | | | |
|---|---|---|---|
| A. | $v_2(\{A,B\}) - v_2(\varnothing)$ | *versus* | $v_1(\{A,B\}) - v_1(\varnothing)$ |
| B. | $v_2(\{A,B\}) - v_2(\{A\})$ | *versus* | $v_1(\{B\}) - v_1(\varnothing)$ |
| C. | $v_2(\{A,B\}) - v_2(\{B\})$ | *versus* | $v_1(\{A\}) - v_1(\varnothing)$ |

AUCTION V: "BIDDER 2'S AUCTION FOR {A}"

An ascending-bid auction is conducted between Bidder 2 and Bidder 1, for each of the following differences in value:

| | | | |
|---|---|---|---|
| A. | $v_2(\{A\}) - v_2(\varnothing)$ | *versus* | $v_1(\{A,B\}) - v_1(\{B\})$ |
| B. | $v_2(\{A\}) - v_2(\{B\})$ | *versus* | $v_1(\{A\}) - v_1(\{B\})$ |

AUCTION VI: "BIDDER 2'S AUCTION FOR {B}"

An ascending-bid auction is conducted between Bidder 2 and Bidder 1, for each of the following differences in value:

| | | | |
|---|---|---|---|
| A. | $v_2(\{B\}) - v_2(\emptyset)$ | *versus* | $v_1(\{A,B\}) - v_1(\{A\})$ |
| B. | $v_2(\{B\}) - v_2(\{A\})$ | *versus* | $v_1(\{B\}) - v_1(\{A\})$ |

Each of the two or three subauctions can be thought of as operating with an ascending clock. The clock begins running at zero, and each bidder simultaneously indicates if she is "in". If both bidders are "in", the auctioneer increments the clock, and again each bidder simultaneously indicates whether she is "in". The subauction concludes at the moment that at most one bidder indicates she is "in", and the outcome of the subauction is described as the price standing on the clock and which (if any) bidder is still "in".

Bidder 1 will be defined to have ***won*** any of the above Auctions I-III when, for each of the subauctions contained in that auction, Bidder 2 did not remain in at the final price on the clock. Similarly, Bidder 2 will be defined to have ***won*** any of the above Auctions IV-VI when, for each of the subauctions contained in that auction, Bidder 1 did not remain in at the final price on the clock. Observe that we will only examine whether the bidder who is named in the title of the auction wins the auction; if she does not, then she will be defined to have ***lost*** the auction.

For example, consider the operation of Auction I, above, if each bidder bids according to her true values as expressed in Eqs. (2.1). For convenience, let us think of the ascending clock as running continuously. We then observe that subauction IA concludes with a "price" of 50 on the clock, and with Bidder 1 remaining "in". We also observe that subauction IB concludes with a "price" of 40 on the clock, and with Bidder 1 remaining "in". Finally, we observe that subauction IC concludes with a "price" of 40 on the clock, and with Bidder 1 remaining "in". Thus, in the terminology we have just defined, Bidder 1 is said to have "won" Auction I.

Table 1A summarizes the outcomes of all the subauctions. Table 1B then summarizes which bidder (if any) won each of the auctions. Table 1B also includes two additional, degenerate auctions whose outcomes are useful to define. We define "Bidder 1's Auction for ∅" to be an auction which Bidder 1 always wins, and we define "Bidder 2's Auction for ∅" to be an auction which Bidder 2 always wins.

Finally, we are ready to define the outcome of the entire procedure of simultaneous auctions and subauctions. Let X denote any subset of the set of available objects, and let ~X = {A,B} $\setminus$ X denote the complement of X. Then the alternative auction procedure concludes with ***X assigned to Bidder 1 and ~X assigned to Bidder 2*** provided that ***Bidder 1 wins Bidder 1's Auction for X*** and ***Bidder 2 wins Bidder 2's Auction for ~X***. In Example 1, observe from Table 1B that the unique X which satisfies this criterion is {A,B}; that is, Bidder 1 wins Bidder 1's Auction for {A,B}, and Bidder 2 wins Bidder 2's Auction for $\varnothing$. Finally, our payment rule shall be that Bidder 1 pays the final price which is reached in Bidder 1's Auction for X, and Bidder 2 pays the final price which is reach in Bidder 2's Auction for ~X. Thus, in Example 1, Bidder 1 pays 50 and Bidder 2 pays 0, fully replicating the outcome of the Vickrey auction.

What is revealed, in the operation of this ascending-bid auction for dissimilar objects? Clearly, the values of Bidder 2 for all of the subsets of the available objects are exposed: Subauction IC reveals that $v_2(\{A\}) = 40$; Subauction IB reveals that $v_2(\{B\}) = 40$; and Subauction IA reveals that $v_2(\{A,B\}) = 50$. However, very minimal information about Bidder 1's values is communicated. Subauction IIA reveals that $v_1(\{A\}) > 10$, but in fact, $v_1(\{A\}) = 200$. Subauction IIIA reveals that $v_1(\{B\}) > 10$, but in fact, $v_1(\{B\}) = 60$. Subauctions IB and IC reveal that $v_1(\{A,B\}) - v_1(\{A\}) > 40$ and $v_1(\{A,B\}) - v_1(\{B\}) > 40$, but in fact, $v_1(\{A,B\}) - v_1(\{A\}) = 60$ while $v_1(\{A,B\}) - v_1(\{B\}) = 200$. The ascending-bid auction design does a reasonable job of maintaining the confidentiality of the high bidder's true valuations.

We shall prove in Section 3 that the above procedure — even when properly generalized to treat arbitrarily many dissimilar objects — is guaranteed to yield an efficient assignment of the objects. Moreover, when bidders' values are in "general position", the termination point of the ascending-bid auction procedure is uniquely determined.

EXAMPLE 2:

Let us next consider a two-object, two-bidder scenario similar to Example 1, only let us modify the values of Eqs. (2.1) so that efficiency now requires each bidder to receive one license. The bidders' values are now given by:

$$(2.2) \qquad \textit{Bidder 1}: \quad \begin{aligned} v_1(\varnothing) &= 0 \\ v_1(\{A\}) &= 200 \\ v_1(\{B\}) &= 30 \\ v_1(\{A,B\}) &= 230 \ ; \end{aligned}$$

$$\textit{Bidder 2}: \quad \begin{aligned} v_2(\varnothing) &= 0 \\ v_2(\{A\}) &= 40 \\ v_2(\{B\}) &= 40 \\ v_2(\{A,B\}) &= 50 \ . \end{aligned}$$

We may again run the six separate, simultaneous auctions listed in Example 1, each again consisting of two or three simultaneous subauctions. The results of the auctions for Example 2 are now somewhat different, as summarized in Tables 2A and 2B.

We see that in Example 2 — unlike Example 1 — Bidder 1 loses Bidder 1's Auction for {A,B}, so the auctioneer will no longer be assigning {A,B} to Bidder 1. However, Bidder 2 now wins Bidder 2's Auction for {B}. Since Bidder 1 continues to win Bidder 1's Auction for {A}, we have shown that {A} now constitutes the set X such that Bidder 1 wins Bidder 1's Auction for X and Bidder 2 wins Bidder 2's Auction for ~X. Finally, observe that Bidder 1's Auction for X stops at a price of 10, which becomes Bidder 1's payment; while Bidder 2's Auction for ~X stops at a price of 30, which becomes Bidder 2's payment. These again match the payments from the Vickrey auction. But, again, the highest values remain reasonably confidential: Auctions IIA and IC have only revealed that $v_1(\{A\}) > 10$ and $v_1(\{A,B\}) - v_1(\{B\}) > 40$, but in fact, $v_1(\{A\}) = v_1(\{A,B\}) - v_1(\{B\}) = 200$. Meanwhile, Auction IB has revealed that $v_2(\{B\}) > 30$, but in fact $v_2(\{B\}) = 40$.

## 3. Results for Two Bidders and Arbitrary Sets of Dissimilar Objects

In this section, we will formulate the two-bidder, ascending-bid auction procedure for arbitrary sets of dissimilar objects. Moreover, we will prove that the auction procedure is guaranteed to yield an efficient assignment of the objects and that, when bidders' values are in "general position", the termination point of the ascending-bid auction procedure is uniquely determined.

Let $\Omega$ denote any finite set of objects which are offered at auction, and let $|\Omega| = M$.

There are two bidders, subscripted by $i$ ($i = 1, 2$). It is assumed that bidders' values for these objects are additively separable from their values for everything else in the world, and that these values are expressible in monetary units. Thus, each bidder, $i$, possesses a value, $v_i(Y)$, for every subset $Y$ of $\Omega$, and we may normalize $v_i(\varnothing) = 0$, for $i = 1, 2$.

DEFINITION 3.1. For each bidder $i = 1, 2$ and for every subset $X \in 2^{\Omega}$ of the available objects, <u>Bidder $i$'s Auction for X</u> is defined to consist of the $2^{M} - 1$ subauctions:

$$v_i(X) - v_i(Y) \qquad \textit{versus} \qquad v_j(\sim Y) - v_j(\sim X) \ ,$$

where $Y \in 2^{\Omega} \setminus X$ and $j \neq i$. Each of the $2^{M} - 1$ subauctions is conducted with an ascending clock. The clock begins with $p_0 = 0$, and then follows an increasing sequence $\{p_t\}$. At each $p_t$, Bidder $i$ must indicate whether $v_i(X) - v_i(Y) > p_t$ and Bidder j must simultaneously indicate whether $v_j(\sim Y) - v_j(\sim X) > p_t$. If both bidders indicate they are "in", the clock increments from $p_t$ to $p_{t+1}$, and the process repeats. If either bidder indicates that she is "not in," the subauction concludes. The outcome of the subauction is described by the final price, $p_t$, on the clock, and which (if any) bidder is still "in" at $p_t$.

DEFINITION 3.2. For each bidder $i = 1, 2$ and for every subset $X \in 2^{\Omega}$ of the available objects, Bidder $i$ will be defined to have <u>won</u> Bidder $i$'s Auction for X if for every $Y \in 2^{\Omega} \setminus X$, the outcome of the subauction corresponding to $Y$ has Bidder $j$ "not in" at the final price.

It should be remarked that, given reasonable conditions on the bidders' values, $v_i(\bullet)$, a number of the $2^{M} - 1$ subauctions included in Definitions 3.1 and 3.2 may be extraneous. For example, suppose that bidders' values are strictly increasing, so that $X \subsetneq Y$ implies $v_i(X) < v_i(Y)$. Then any subauction in which $v_i(X) - v_i(Y)$ is compared to $v_j(\sim Y) - v_j(\sim X)$, where $X \subsetneq Y$, is certain to be extraneous, since $v_i(X) - v_i(Y) < 0$ and $v_j(\sim Y) - v_j(\sim X) < 0$, so the outcome of the auction will have a price of zero, and Bidder $j$ (as well as Bidder $i$) will be "not in."

It should also be remarked that, if instead of starting the clock for each subauction at zero, we had started the clock at $-\infty$, it would have been unnecessary to run both Bidder $i$'s Auction for $X$ and Bidder $j$'s Auction for $\sim X$. The two auctions test the same inequalities, and therefore one or the other would suffice. However, precisely because we are starting the clock for each subauction at zero, both Bidder *i*'s Auction for $X$ and Bidder *j*'s Auction for $\sim X$ are necessary to extract all of the information necessary for efficient assignment from the bidders.

The following lemmas and theorems will assume that the clock <u>runs continuously</u>, i.e., we make the approximation that time, $t$, is continuous rather than discrete, and that $p_t$ is a continuous, increasing function with the properties that $p_0 = 0$ and $\lim_{t \uparrow \infty} p_t = \infty$. Observe that any of the results below will be closely approximated in situations where the clock runs discretely, but the width of the price grid is small.

LEMMA 3.3. Consider any bidder $i = 1, 2$ and any subset $X \in 2^{\Omega}$ of the available objects. If the clock is run continuously and if bidders bid sincerely (i.e., if their response to every question asked in every subauction is truthful), then Bidder $i$ wins Bidder $i$'s Auction for $X$ if and only if, for every $Y \in 2^{\Omega} \setminus X$:

(i) $\quad v_i(X) - v_i(Y) \;\geq\; v_j(\sim Y) - v_j(\sim X)$ ,

or:

(ii) $\quad v_j(\sim Y) - v_j(\sim X) \;\leq\; 0$ .

PROOF. Follows immediately from Definitions 3.1 and 3.2.

The following theorem is the main result of this section:

THEOREM 3.4. The assignment of $X \in 2^{\Omega}$ to Bidder 1 and $\sim X$ to Bidder 2 is an efficient allocation if and only if Bidder 1 wins Bidder 1's Auction for $X$ and Bidder 2 wins Bidder 2's Auction for $\sim X$.

PROOF. Suppose that Bidder 1 wins Bidder 1's Auction for $X$ and Bidder 2 wins Bidder 2's Auction for $\sim X$. Then, by Lemma 3.3, for every $Y \in 2^{\Omega} \setminus X$, inequality (3.1) or (3.2) holds, and inequality (3.3) or (3.4) holds, defined as follows:

(3.1) $\quad v_1(X) - v_1(Y) \;\geq\; v_2(\sim Y) - v_2(\sim X) \;>\; 0$ ,

(3.2) $\quad v_2(\sim Y) - v_2(\sim X) \;\leq\; 0$ , (3.3) $\quad v_2(\sim X) - v_2(\sim Y) \ \geq \ v_1(Y) - v_1(X) \ > \ 0$ ,

(3.4) $\quad v_1(Y) - v_1(X) \ \leq \ 0$ .

Given any $Y \in 2^{\Omega} \setminus X$, if Ineq. (3.1) or (3.3) holds, simple rearranging of terms implies that:

(3.5) $\quad v_1(X) + v_2(\sim X) \ \geq \ v_1(Y) + v_2(\sim Y)$ .

If neither Ineq. (3.1) nor Ineq. (3.3) holds, then both Ineqs. (3.2) and (3.4) hold. By (3.4), $v_1(X) - v_1(Y) \geq 0$. Combining this inequality with Ineq. (3.2) implies: $v_1(X) - v_1(Y) \geq v_2(\sim Y) - v_2(\sim X)$. Simple rearranging of terms again yields Ineq. (3.5).

We thus conclude that Ineq. (3.5) holds for all $Y \in 2^{\Omega} \setminus X$, i.e., the assignment of $X$ to Bidder 1 and $\sim X$ to Bidder 2 is efficient.

Conversely, suppose that the assignment of $X$ to Bidder 1 and $\sim X$ to Bidder 2 is efficient, i.e., suppose that Ineq. (3.5) holds for all $Y \in 2^{\Omega} \setminus X$. Then for each $Y \in 2^{\Omega} \setminus X$, there are two cases to consider:

***Case I*:** $v_2(\sim Y) - v_2(\sim X) \ > \ 0$.

By Ineq. (3.5), observe that $v_1(X) - v_1(Y) \geq v_2(\sim Y) - v_2(\sim X)$. Hence, Ineq. (3.1) holds, and so the outcome of the subauction involving $Y$ is consistent with Bidder 1 winning Bidder 1's Auction for $X$. Also observe that $v_1(Y) - v_1(X) < 0$, and so the outcome of the subauction on $\sim Y$ is consistent with Bidder 2 winning Bidder 2's Auction for $\sim X$.

***Case II*:** $v_2(\sim Y) - v_2(\sim X) \ \leq \ 0$.

Observe immediately that the outcome of the subauction involving $Y$ is consistent with Bidder 1 winning Bidder 1's Auction for $X$. Also observe by Ineq. (3.5) that $v_2(\sim X) - v_2(\sim Y) \geq v_1(Y) - v_1(X)$, and so the outcome of the subauction on $\sim Y$ is consistent with Bidder 2 winning Bidder 2's Auction for $\sim X$.

The above statements hold for all $Y \in 2^{\Omega} \setminus X$ (and, hence, also for all $\sim Y \in 2^{\Omega} \setminus \sim X$), so we conclude that Bidder 1 wins Bidder 1's Auction for $X$ and Bidder 2 wins Bidder 2's Auction for $\sim X$, as we desired to prove. □

DEFINITION 3.5. We will say that $v_1(\bullet)$ and $v_2(\bullet)$ are in ***general position*** if for every $X_1, X_2, Y_1, Y_2 \in 2^{\Omega}$ such that $X_1 \neq Y_1$ and $X_2 \neq Y_2$, we have:

$$v_1(X_1) - v_1(Y_1) \neq v_2(X_2) - v_2(Y_2) \quad .$$

COROLLARY 3.6. There always exists at least one $X \in 2^{\Omega}$ such that Bidder 1 wins Bidder 1's Auction for $X$ and Bidder 2 wins Bidder 2's Auction for $\sim X$.

If $v_1(\bullet)$ and $v_2(\bullet)$ are in general position, then there exists one and only $X \in 2^{\Omega}$ such that Bidder 1 wins Bidder 1's Auction for $X$ and Bidder 2 wins Bidder 2's Auction for $\sim X$. Moreover, the assignment of $X$ to Bidder 1 and $\sim X$ to Bidder 2 is the unique efficient allocation.

PROOF. There exists at least one solution to the problem: $\max_{X \in 2^{\Omega}} \{v_1(X) + v_2(\sim X)\}$. By Theorem 3.4, if $X$ is such a solution, then Bidder 1 wins Bidder 1's Auction for $X$ and Bidder 2 wins Bidder 2's Auction for $\sim X$.

Suppose that $X$ and $X'$ are both solutions. Then:

$$v_1(X) + v_2(\sim X) = v_1(X') + v_2(\sim X') \quad ,$$

and so:

$$v_1(X) - v_1(X') = v_2(\sim X') - v_2(\sim X) \quad ,$$

implying that $v_1(\bullet)$ and $v_2(\bullet)$ are not in general position, unless $X = X'$. □

## 4. An Example with Three Bidders and Two Dissimilar Objects

The results of Section 3 completely treat the case of two bidders and an arbitrary number of dissimilar objects. In order to gain some insight into the treatment of more than two bidders, let us extend Example 2 by adding an additional bidder.

EXAMPLE 3:

(4.1) *Bidder 1*:
$$\begin{aligned} v_1(\varnothing) &= 0 \\ v_1(\{A\}) &= 200 \\ v_1(\{B\}) &= 30 \\ v_1(\{A,B\}) &= 230 \ ; \end{aligned}$$

*Bidder 2*:
$$\begin{aligned} v_2(\varnothing) &= 0 \\ v_2(\{A\}) &= 40 \\ v_2(\{B\}) &= 40 \\ v_2(\{A,B\}) &= 50 \ ; \end{aligned}$$

*Bidder 3*:
$$\begin{aligned} v_3(\varnothing) &= 0 \\ v_3(\{A\}) &= 25 \\ v_3(\{B\}) &= 75 \\ v_3(\{A,B\}) &= 125 \ . \end{aligned}$$

The basic ingredient of the ascending-bid auction for three bidders is to first consider the bidders pairwise. For each pair $(j,k)$ of bidders, and for each subset $W$ of the set of available objects, we conduct a "virtual auction" according to the procedure of Section 3, above. The outcomes of these virtual auctions provide the efficient assignment of the objects in $W$ — if they were to be allocated only among Bidders $j$ and $k$ — as well as lower bounds on the values associated with the efficient assignments. (As we will see later, if more precise bounds are needed concerning the values associated with the efficient assignments, the virtual auctions can be restarted.) Second, we conduct another series of virtual auctions, which this time place Bidder $i$ in competition with the combination of Bidders $j$ and $k$, again using the procedure of Section 3. When the second series of virtual auctions is completed, the auctioneer has elicited the efficient allocation among Bidders $i$, $j$, and $k$, as well as how much of a payment to assess Bidder $i$. Performing these steps, for each of $i = 1, 2, 3$, the auctioneer obtains all of the information needed to implement the outcome rule of the Vickrey auction. However, since the procedure of Section 3 was followed, the auctioneer avoids being unnecessarily intrusive in eliciting the values of the high bidders, providing the same advantages as before.

For each pair $(j,k)$ of bidders, and for each subset $W$ of the set of available objects, let $v_{jk}(\mathrm{W})$ denote the total value if the objects in $W$ are allocated efficiently between Bidders $j$ and $k$. Observe that Bidders 1 and 2 of Example 3 are exactly the same as in Example 2, so we have already analyzed the virtual auction used to construct $v_{12}(\{\mathrm{A,B}\})$, in Section 2 and Tables 2A and B. Thus, if the objects can only be allocated between Bidder 1 and Bidder 2, the efficient assignment is to give Object A to Bidder 1 and Object B to Bidder 2, and:

(4.2) $$v_{12}(\{\mathrm{A,B}\}) = v_1(\mathrm{A}) + v_2(\mathrm{B}) \quad .$$

Similarly, the general procedure for two bidders trivially yields:

(4.3) $$v_{12}(\{\mathrm{A}\}) = v_1(\{\mathrm{A}\}) \quad ,$$

and:

(4.4) $$v_{12}(\{\mathrm{B}\}) = v_2(\{\mathrm{B}\}) \quad .$$

The second step of the procedure is then to treat the combination of Bidders 1 and 2 as an artificial bidder (who is denoted by “12” and whose values for {A,B}, {A}, and {B} are given by Eqs. (4.2), (4.3) and (4.4), respectively), and to run a virtual auction between Bidder 3 and the combination Bidder “12”. The only obstacle in executing this program is that some of the subauctions require the combination bidder to report information such as whether $v_{12}(\{\mathrm{A}\}) - v_{12}(\{\mathrm{B}\}) > \mathrm{p}$, which is equivalent to reporting whether $v_1(\{\mathrm{A}\}) - v_2(\{\mathrm{B}\}) > \mathrm{p}$, but answering this question requires using information part of which is known only by Bidder 1 and part of which is known only by Bidder 2. The auctioneer deals with this difficulty by simultaneously running ascending clocks for each of $v_1(\{\mathrm{A}\})$ and $v_2(\{\mathrm{B}\})$, i.e., asking Bidder 1 whether $v_1(\{\mathrm{A}\}) > p_t$, and asking Bidder 2 whether $v_2(\{\mathrm{B}\}) > p_t$, for gradually-incrementing $p_t$. If, for example, the clock for $v_2(\{\mathrm{B}\})$ stops first, then the auctioneer has ascertained that $v_1(\{\mathrm{A}\}) - v_2(\{\mathrm{B}\}) > 0$, and by continuing to increment $p_t$, the auctioneer learns whether $v_1(\{\mathrm{A}\}) - v_2(\{\mathrm{B}\}) > p_t - v_2(\{\mathrm{B}\})$.

Table 3A summarizes the outcomes of all the subauctions between Bidder 3 and combination bidder "12", and Table 3B then summarizes whether the named bidder won her auction. Recall, by Corollary 3.5, that there is guaranteed to exist one and only one $X$ such that Bidder 3 wins Bidder 3's Auction for $X$ and Bidder "12" wins Bidder 12's Auction for $\sim X$. In this case, $X = \{B\}$, so Object B is assigned to Bidder 3 and Object A is assigned to the combination Bidder "12" (and, hence, to Bidder 1). Moreover, Bidder 3's Auction for {B} stopped at a price of 40, so Bidder 3 pays 40 for Object B; and Bidder 12's Auction for {A} stopped at a price of 50, so Bidder 1 pays at least 50 for Object A.[3,4]

What information is elicited in the course of the above three-bidder procedure? We already know (from Example 2) that in the first step of the procedure, it was revealed that $v_1(\{A\}) > 10$, $v_1(\{A,B\}) - v_1(\{B\}) > 40$, and $v_2(\{B\}) > 30$. In the second step of the procedure, as remarked above, it became necessary to establish that $v_1(\{A\}) - v_2(\{B\}) > 0$. In running the clock on $v_2(\{B\})$, it was revealed that $v_2(\{B\}) = 40$ and that $v_1(\{A\}) > 40$. Then, in the face-off between Bidder 3 and combination Bidder "12", it was further revealed that $v_1(\{A\}) > 50$ and $v_3(\{B\}) > 40$. However, the high values — the facts that $v_1(\{A\}) = 200$ and $v_3(\{B\}) = 75$ — are still kept nicely confidential in the three-bidder procedure.

## 5. Results for *n* Bidders and Arbitrary Sets of Dissimilar Objects

We now give general results for $n$ bidders and arbitrary sets of dissimilar objects. We begin by defining $N = \{1, 2, ..., n\}$ to be the set of all bidders. The general procedure is:

1. For any $j \in N$ and for any $k \in N \setminus \{j\}$, run the two-bidder auction of Section 3 between Bidder $j$ and Bidder $k$, for every $W \in 2^{\Omega} \setminus \varnothing$. Use the results to define composite Bidder "*jk*".

---

[3] The correct way to calculate Bidder 1's payment for Object A is to perform the following steps. First, run the virtual auctions between Bidder 2 and Bidder 3 for {A,B}, {A}, and {B}. Second, run the virtual auction between Bidder 1 and combination Bidder "23" for {A,B}. In the latter step, Bidder 1's Auction for {A} stops at a price of 50, so we conclude that 50 is the correct payment for Bidder 1.

[4] It would also be possible to run the virtual auctions between Bidder 1 and Bidder 3 for {A,B}, {A}, and {B}; and then run the virtual auction between Bidder 2 and combination Bidder "13" for {A,B}. However, this is unnecessary here. We already know that Bidder 2 is not assigned any objects, and therefore we know immediately that her payment is zero.

2. For any $i \in N \setminus \{j,k\}$, run the two-bidder auction of Section 3 between Bidder *i* and composite Bidder "*jk*", for every $W \in 2^{\Omega} \setminus \varnothing$. As needed, return to the bidders of previous steps and restart the associated ascending clocks to elicit additional information. Use the results to define composite Bidder "*ijk*".

3. For any $h \in N \setminus \{i,j,k\}$, run the two-bidder auction of Section 3 between Bidder *h* and composite Bidder "*ijk*", for every $W \in 2^{\Omega} \setminus \varnothing$. As needed, return to the bidders of previous steps and restart the associated ascending clocks to elicit additional information. Use the results to define composite Bidder "*hijk*".

. . .

*n*. For the one remaining $r \in N$, run the two-bidder auction of Section 3 between Bidder *r* and composite Bidder "$N \setminus \{r\}$". As needed, return to the bidders of previous steps and restart the associated ascending clocks to elicit additional information. If, in the outcome of this final auction, Bidder *r* wins Bidder *r*'s Auction for *X*, and composite Bidder "$N \setminus \{r\}$" wins the Auction of Bidder "$N \setminus \{r\}$" for ~*X*, then *X* is the subset of objects assigned to Bidder *r*. Moreover, if *p* is the highest price reached in Bidder *r*'s Auction for *X*, then *p* is the payment charged to Bidder *r*.

Observe that it is (fortunately) unnecessary to run through the above steps for every permutation of *N*. Computations can be vastly reduced by re-using intermediate steps. For example, in the six-bidder auction, it is useful to first generate the composite bidder "1234". Then, one conducts additional auctions with the sequencing of Bidder 5 followed by Bidder 6, in order to obtain the assignment and payment of Bidder 6; and then with the sequencing of Bidder 6 followed by Bidder 5, in order to obtain the assignment and payment of Bidder 5. Another shortcut is available by recognizing that any bidder who receives an assignment of the empty set must necessarily be assessed a payment of zero.

The results on the general procedure are straightforward extensions of the results of Section 3. We have:

THEOREM 5.1. For any step of the general procedure, for any $i \in N$, and for any $J$ subset $N \setminus \{i\}$, the assignment of $X \in 2^{\Omega}$ to Bidder $i$ and $\sim X$ to composite Bidder J is an efficient allocation if and only if Bidder $i$ wins Bidder $i$'s Auction for $X$ and composite Bidder $J$ wins composite Bidder $J$'s Auction for $\sim X$.

PROOF. Same as proof of Theorem 3.4.

COROLLARY 5.2. For any step of the general procedure, for any $i \in N$, and for any $J$ subset $N \setminus \{i\}$, there always exists at least one $X \in 2^{\Omega}$ such that Bidder $i$ wins Bidder $i$'s Auction for $X$ and composite Bidder $J$ wins composite Bidder $J$'s Auction for $\sim X$.

If $v_i(\bullet)$ and $v_J(\bullet)$ are in general position, then there exists one and only $X \in 2^{\Omega}$ such that Bidder $i$ wins Bidder $i$'s Auction for $X$ and composite Bidder $J$ wins composite Bidder $J$'s Auction for $\sim X$. Moreover, the assignment of $X$ to Bidder $i$ and $\sim X$ to composite Bidder $J$ is the unique efficient allocation.

PROOF. Same as proof of Corollary 3.6.

Theorem 5.1 and Corollary 5.2 establish that an outcome is yielded at every step of the general procedure, and that the outcome is consistent with full efficiency.

## 6. The Submission of Bids

The most straightforward way to implement the auction proposed in this paper is by use of what might be referred to as a "safe bidding terminal." The most basic version of a safe bidding terminal would operate as follows. Before the start of the auction, each bidder $i$ enters her value, $v_i(W)$, for every subset $W \in 2^{\Omega}$ of the available objects. Once the auction begins, these values are locked in and may not be changed — quite like a submission of sealed bids. However, the values, $v_i(W)$, reside only in the memory of the safe bidding terminal, and are never directly transmitted to the auctioneer. Instead, the auction is conducted by the auctioneer's "central computer" sending a series of questions to bidders' terminals, each question of the form: Is $v_i(X) - v_i(Y) > p$ ? ($X$ and $Y$ are subsets of $\Omega$, and $p$ is a nonnegative number.) Bidder $i$'s safe bidding terminal automatically responds to these questions, on behalf of Bidder $i$, by using the values which Bidder $i$ entered before the start of the auction. The responses to all of the

questions become known to the auctioneer (and, of course, are used in determining the auction outcome). However, when the auctioneer's central computer sends the signal that the auction has concluded, the bidder's values are erased from the safe bidding terminal's memory.
*Any information which was not elicited by the auctioneer's questions remains confidential.*

If bidders' values are in general position, and if the inquiry prices are increased in sufficiently small increments, we have seen that the general auction procedure leads to the unique allocation and payments of the Vickrey auction. With complete information or with independent private values, it immediately follows that sincere bidding is a dominant strategy. Observe that this statement holds true regardless of the pacing of the auction (as discussed in the next Section), as the pacing rules have no effect on the auction-determined allocation and payments. However, the pacing of the auction will affect the precise questions which are directed to bidders, and so the pacing rules will affect which information is elicited in the auction and which information remains confidential.

An alternative way of conducting the auction is to allow bidders to adjust their values as the auction proceeds. However, observe that the adjustment of reported values, if allowed in unrestricted form, may lead to the answering of new questions in ways which contradict the answers to old questions. Thus, if bidders are permitted to change their reported values, they should probably be restricted to only making changes which are consistent with all answers that they have previously provided.

There are at least two ways to allow bidders to adjust their reported values midstream. The first way is to include periodic breaks in the auction activity, and allow bidders to manually make changes which are consistent will all prior answers generated. The second way is to specifically construct the safe bidding terminal so as to admit *contingent reports of values*. That is, the information which the bidder enters into the terminal provides a strategy whereby values may be increased or decreased, depending on the actual bid history which is generated in the auction.

Allowing adjustments to reported values — by either means — is largely irrelevant, under complete information or independent private values. However, if bidders' signals are affiliated, then there are potential gains associated with allowing bidders to increase or decrease their reported values depending on the bidding activities of other auction participants, as in Milgrom and Weber (1982).

## 7. Applications

Many applications of the efficient ascending-bid auction for dissimilar objects may seem inordinately cumbersome, on account that, with $M$ objects, each bidder is required to determine a valuation for each of $2^{M-1}$ subsets of objects. However, let me now briefly describe one example of a potential application where the operation of the auction could be quite straightforward. Suppose that the Government wished to auction a collection of $M$ television licenses in a city, and the Government enforced a regulation limiting each buyer to holding at most one television license in the city. Observe that it is probably sensible to view this as a dissimilar-object auction, since (at least with current technology), some television frequencies are more desirable than others. Moreover, in a larger setting, bidders may value different television channels differently, depending on what channel a given bidder already holds in other cities.

In this situation, it would only be necessary for each bidder to determine a valuation for the $M$ feasible subsets of licenses: namely, the set of $M$ singletons. Moreover, the collection of subauctions which would need to be considered between various pairs of bidders would now be comparatively small. Given the considerations discussed earlier in the paper, the efficient ascending-bid auction for dissimilar objects may be an attractive candidate for this application.

## SUMMARY OF RESULTS FOR EXAMPLE 1

| Subauction | Final Price | Bidder 1 In? | Bidder 2 In? |
|---|---|---|---|
| IA | 50 | Yes | No |
| IB | 40 | Yes | No |
| IC | 40 | Yes | No |
| IIA | 10 | Yes | No |
| IIB | 0 | Yes | No |
| IIIA | 10 | Yes | No |
| IIIB | 0 | Yes | No |
| IVA | 50 | Yes | No |
| IVB | 10 | Yes | No |
| IVC | 10 | Yes | No |
| VA | 40 | Yes | No |
| VB | 0 | Yes | No |
| VIA | 40 | Yes | No |
| VIB | 0 | No | No |

TABLE 1A

| Auction | Description of Auction | Does Named Bidder Win? |
|---|---|---|
| I | Bidder 1's Auction for {A,B} | Bidder 1 Wins |
| II | Bidder 1's Auction for {A} | Bidder 1 Wins |
| III | Bidder 1's Auction for {B} | Bidder 1 Wins |
| — | Bidder 1's Auction for $\emptyset$ | Bidder 1 Wins |
| IV | Bidder 2's Auction for {A,B} | Bidder 2 Loses |
| V | Bidder 2's Auction for {A} | Bidder 2 Loses |
| VI | Bidder 2's Auction for {B} | Bidder 2 Loses |
| — | Bidder 2's Auction for $\emptyset$ | Bidder 2 Wins |

TABLE 1B

# SUMMARY OF RESULTS FOR EXAMPLE 2

| Subauction | Final Price | Bidder 1 In? | Bidder 2 In? |
|---|---|---|---|
| IA | 50 | Yes | No |
| IB | 30 | No | Yes |
| IC | 40 | Yes | No |
| IIA | 10 | Yes | No |
| IIB | 0 | Yes | No |
| IIIA | 10 | Yes | No |
| IIIB | 0 | Yes | No |
| IVA | 50 | Yes | No |
| IVB | 10 | Yes | No |
| IVC | 10 | Yes | No |
| VA | 40 | Yes | No |
| VB | 0 | Yes | No |
| VIA | 30 | No | Yes |
| VIB | 0 | No | No |

TABLE 2A

| Auction | Description of Auction | Does Named Bidder Win? |
|---|---|---|
| I | Bidder 1's Auction for {A,B} | Bidder 1 Loses |
| II | Bidder 1's Auction for {A} | Bidder 1 Wins |
| III | Bidder 1's Auction for {B} | Bidder 1 Wins |
| — | Bidder 1's Auction for ∅ | Bidder 1 Wins |
| IV | Bidder 2's Auction for {A,B} | Bidder 2 Loses |
| V | Bidder 2's Auction for {A} | Bidder 2 Loses |
| VI | Bidder 2's Auction for {B} | Bidder 2 Wins |
| — | Bidder 2's Auction for ∅ | Bidder 2 Wins |

TABLE 2B

## SUMMARY OF RESULTS FOR EXAMPLE 3

| Subauction | Final Price | Bidder 3 In? | Bidder “12” In? |
|---|---|---|---|
| IA | 125 | No | Yes |
| IB | 40 | Yes | No |
| IC | 50 | No | Yes |
| IIA | 25 | No | Yes |
| IIB | 0 | No | Yes |
| IIIA | 40 | Yes | No |
| IIIB | 0 | Yes | No |
| IVA | 125 | Yes | No |
| IVB | 40 | No | Yes |
| IVC | 25 | Yes | No |
| VA | 50 | No | Yes |
| VB | 0 | No | Yes |
| VIA | 40 | Yes | No |
| VIB | 0 | Yes | No |

TABLE 3A

| Auction | Description of Auction | Does Named Bidder Win? |
|---|---|---|
| I | Bidder 3’s Auction for {A,B} | Bidder 3 Loses |
| II | Bidder 3’s Auction for {A} | Bidder 3 Loses |
| III | Bidder 3’s Auction for {B} | Bidder 3 Wins |
| — | Bidder 3’s Auction for ∅ | Bidder 3 Wins |
| IV | Bidder “12” Auction for {A,B} | Bidder “12” Loses |
| V | Bidder “12” Auction for {A} | Bidder “12” Wins |
| VI | Bidder “12” Auction for {B} | Bidder “12” Loses |
| — | Bidder “12” Auction for ∅ | Bidder “12” Wins |

TABLE 3B